Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS
Autor: Shastri, Parvati
Kapitel: §5. Main Theorem and examples
DOI: https://doi.org/10.5169/seals-60415

repeating the above argument we obtain a similar decomposition of $N_1 : N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_j is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. Main Theorem and examples

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. Proposition. *Let L be a unimodular* **Z***-lattice of type $n\mathrm{D}_4$ such that $\mathscr{H}^n \subset L \subset \mathscr{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \operatorname{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on $\mathscr{H}^{*n}$, where δ_i is the isometry on $\mathscr{H}^*$ given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ.*

Proof. Let σ be a perfect isometry of $(L, Tr \circ h)$. Then σ induces an automorphism of $\mathscr{H}^n$ and extends naturally to a perfect isometry of $\mathscr{H}^{*n}$. In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of $\mathbf{F}_4^n$, leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α. Let δ_i denote left multiplication on $\mathscr{H}^*$ by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \operatorname{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since δ induces an isometry of $\mathscr{H}^{*n}$ which fixes $\mathscr{H}^n$ and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. Theorem. *Let (L, S) be an unimodular* **Z***-lattice of type $n\mathrm{D}_4$. Then, L has a perfect isometry if and only if there exists an $\mathscr{H}$-lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every $\mathscr{H}$-lattice admits a perfect isometry (3.2). Conversely let (L, S) be a **Z**-lattice of type $n\mathrm{D}_4$, which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \dots, n\}$ such that L is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f : \mathscr{H}^n \to \mathscr{H}^n$ be defined by $f = \operatorname{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \mathrm{id}$ for $i \in T$ and $f_i =$ the involution on $\mathscr{H}$ for $i \notin T$. Then it is easy to check that f is an isometry of $(L, Tr \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i) .$$

Clearly L' is invariant under left multiplication by ξ. Further, since $\mathscr{P}L' \subseteq \mathscr{P}\mathscr{H}^{*n} \subseteq \mathscr{H}^n \subseteq L'$, it follows that L' is an $\mathscr{H}$-lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type D_4.

5.3. THEOREM. *Let (L, S), be a positive definite unimodular symmetric bilinear space over $\mathbf{Z}$, of rank n. Suppose that the set of vectors of norm 2 form a root system of type*

$$R = \underset{1 \leqslant i \leqslant p}{\perp} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

with, $\sum_{1 \leqslant i \leqslant p} 2k_i + 6q + 8r + 4s = n$. *Then the following hold:*

(i) *The $\mathbf{Z}$-lattice L decomposes as $L = L_1 \perp L_2 \perp L_3$, where each L_i is unimodular, with asociated root systems of type $R_1 = \underset{1 \leqslant i \leqslant p}{\perp} A_{2k_i} \perp qE_6$, $R_2 = rE_8$, $R_3 = sD_4$, respectively.*

(ii) *The $\mathbf{Z}$-lattice L admits a perfect isometry if and only if L_3 is isometric to the trace form of an $\mathscr{H}$-lattice.*

(iii) *If L admits a perfect isometry, then it admits a perfect isometry σ such that the induced map $\eta(\sigma)$ on $\mathbf{Z}R^{\#}/\mathbf{Z}R$, corresponds to multiplication by -1, on the components corresponding to A_{2k_i}, E_6, and E_8, and to multiplication by ω, on the components corresponding to D_4.*

Proof. (i) Since E_8 is unimodular, it is clear that $L = L_2 \perp K$, where $L_2 \simeq r\mathbf{Z}E_8$, and K is unimodular with associated root system of type $R_1 \perp R_3$. So to prove (i), it is enough to prove that K decomposes as $L_1 \perp L_3$. This would follow if we show that $\eta(K)$ decomposes as, $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3)$.

Let $z = (x, y) \in \eta(K)$, with x in $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ and y in $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3$. Since $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ is a group of exponent $3 \cdot \prod_{1 \leqslant i \leqslant p} (2k_i + 1)$, and $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3 \simeq \mathbf{F}_4^m$, it follows that, $(0, y) = 3\left(\prod_{1 \leqslant i \leqslant p} (2k_i + 1)\right)z \in \eta(K)$. Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

5.4. *Examples.* We conclude this section by giving some examples of $\mathscr{H}$-lattices of type nD_4 as well as $\mathbf{Z}$-lattices of type nD_4 which are not $\mathscr{H}$-lattices. Let $\{e_k\}_{1 \leqslant k \leqslant n}$ denote the standard $\mathscr{H}$-basis of $\mathscr{H}^n$. We consider two cases. For $n = 4m$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \leqslant j \leqslant 2m - 2$, and

$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}$. For $n = 4m + 2$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k, 0 \leqslant j \leqslant 2m - 1$, and $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$. Let $\lambda = 1/1 + i$ and let L_n be the $\mathscr{H}$-lattice generated by $\mathscr{H}^n \cup \{\lambda\varepsilon_1, \lambda\varepsilon_2, ..., \lambda\varepsilon_{n/2}\}$. In view of [M-O-S], $\eta(L)$ is a maximal totally isotropic subspace of $\mathbf{F}_4^n$, and every vector $x \in \eta(L)$ has at least four nonzero coordinates. Since $Tr \circ h(x, x) \geqslant 1$, for every x belonging to $\mathscr{H}^*$, it follows easily that the set of vectors of norm 2 in L_n is $n\mathrm{D}_4$. Clearly L_n is unimodular.

For $n = 6$, this gives the unique unimodular $\mathbf{Z}$-lattice of type $6\mathrm{D}_4$ which is also an $\mathscr{H}$-lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable $\mathbf{Z}$-lattices of type $n\mathrm{D}_4$ for $n \leqslant 14$, which are $\mathscr{H}$-lattices. The following construction gives an example of a $\mathbf{Z}$-lattice of type $8\mathrm{D}_4$ which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ which is not an $\mathscr{H}$-lattice is 32). For $1 \leqslant k \leqslant 8$, let ρ_k be equal to ξ if k is even and let ρ_k be equal to 1 if k is odd. Let $\beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i$, $\beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$ for $n \leqslant j \leqslant 2$, $\beta_7 = \xi \,.\, \sum_{i=1}^{4} e_{2i}$ and $\beta_8 = \bar{\xi} \,.\, \sum_{i=1}^{4} e_{2i-1}$. Let Λ be the $\mathbf{Z}$-linear subspace of $\mathscr{H}^{*8}$ spanned by $\mathscr{H}^8$ and $\{\lambda\beta_i\}_{1 \leqslant i \leqslant 8}$. Then $\eta(\Lambda)$ is a maximal totally isotropic subspace of $(\mathbf{F}_4^8, Tr \circ \eta(h))$. It can be easily checked that Λ is a $\mathbf{Z}$-lattice of type $8\mathrm{D}_4$. Further $\eta(\Lambda)$ is not invariant under $\mathrm{diag}(\alpha_1, ..., \alpha_i, ..., \alpha_8)$ for any choice of α_i in $\{\omega, \omega^2\}$. Thus in view of Proposition 4.7, the lattice Λ does not admit any perfect isometry. The above construction easily generalizes to give a family of $\mathbf{Z}$-lattices Λ_{4n} of dimension $16m$, $m \geqslant 2$, which are not $\mathscr{H}$-lattices.

ACKNOWLEDGEMENT. I thank Eva Bayer for critically going through an earlier version of the manuscript and for making useful comments, which led to a better exposition of this work. My thanks are also due to my teachers R. Parimala and R. Sridharan who showed deep interest in my work. I thank H.G. Quebbemann for carefully going through the manuscript and making valuable suggestion.