

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	39 (1993)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS
Autor:	Shastri, Parvati
Kapitel:	§5. Main Theorem and examples
DOI:	https://doi.org/10.5169/seals-60415

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

repeating the above argument we obtain a similar decomposition of $N_1: N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_j is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. PROPOSITION. *Let L be a unimodular \mathbf{Z} -lattice of type nD_4 such that $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on \mathcal{H}^{*n} , where δ_i is the isometry on \mathcal{H}^* given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ .*

Proof. Let σ be a perfect isometry of $(L, Tr \circ h)$. Then σ induces an automorphism of \mathcal{H}^n and extends naturally to a perfect isometry of \mathcal{H}^{*n} . In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of F_4^n , leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α . Let δ_i denote left multiplication on \mathcal{H}^* by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since δ induces an isometry of \mathcal{H}^{*n} which fixes \mathcal{H}^n and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. THEOREM. *Let (L, S) be an unimodular \mathbf{Z} -lattice of type nD_4 . Then, L has a perfect isometry if and only if there exists an \mathcal{H} -lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every \mathcal{H} -lattice admits a perfect isometry (3.2). Conversely let (L, S) be a \mathbf{Z} -lattice of type nD_4 , which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \dots, n\}$ such that L is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$ be defined by $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \text{id}$ for $i \in T$ and $f_i =$ the involution on \mathcal{H} for $i \notin T$. Then it is easy to check that f is an isometry of $(L, Tr \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$

Clearly L' is invariant under left multiplication by ξ . Further, since $\mathcal{P}L' \subseteq \mathcal{PH}^{*n} \subseteq \mathcal{H}^n \subseteq L'$, it follows that L' is an \mathcal{H} -lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type D_4 .

5.3. THEOREM. *Let (L, S) , be a positive definite unimodular symmetric bilinear space over \mathbf{Z} , of rank n . Suppose that the set of vectors of norm 2 form a root system of type*

$$R = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

with, $\sum_{1 \leq i \leq p} 2k_i + 6q + 8r + 4s = n$. Then the following hold:

(i) *The \mathbf{Z} -lattice L decomposes as $L = L_1 \perp L_2 \perp L_3$, where each L_i is unimodular, with associated root systems of type $R_i = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6$,*

$R_2 = rE_8$, $R_3 = sD_4$, respectively.

(ii) *The \mathbf{Z} -lattice L admits a perfect isometry if and only if L_3 is isometric to the trace form of an \mathcal{H} -lattice.*

(iii) *If L admits a perfect isometry, then it admits a perfect isometry σ such that the induced map $\eta(\sigma)$ on $\mathbf{Z}R^{\#}/\mathbf{Z}R$, corresponds to multiplication by -1 , on the components corresponding to A_{2k_i} , E_6 , and E_8 , and to multiplication by ω , on the components corresponding to D_4 .*

Proof. (i) Since E_8 is unimodular, it is clear that $L = L_2 \perp K$, where $L_2 \simeq r\mathbf{Z}E_8$, and K is unimodular with associated root system of type $R_1 \perp R_3$. So to prove (i), it is enough to prove that K decomposes as $L_1 \perp L_3$. This would follow if we show that $\eta(K)$ decomposes as, $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3)$.

Let $z = (x, y) \in \eta(K)$, with x in $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ and y in $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3$. Since $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ is a group of exponent 3. $\prod_{1 \leq i \leq p} (2k_i + 1)$, and $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3 \simeq F_4^m$,

it follows that, $(0, y) = 3(\prod_{1 \leq i \leq p} (2k_i + 1))z \in \eta(K)$. Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

5.4. Examples. We conclude this section by giving some examples of \mathcal{H} -lattices of type nD_4 as well as \mathbf{Z} -lattices of type nD_4 which are not \mathcal{H} -lattices. Let $\{e_k\}_{1 \leq k \leq n}$ denote the standard \mathcal{H} -basis of \mathcal{H}^n . We

consider two cases. For $n = 4m$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \leq j \leq 2m-2$, and

$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}$. For $n = 4m+2$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \leq j \leq 2m-1$,

and $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$. Let $\lambda = 1/1 + i$ and let L_n be the \mathcal{H} -lattice generated by $\mathcal{H}^n \cup \{\lambda \varepsilon_1, \lambda \varepsilon_2, \dots, \lambda \varepsilon_{n/2}\}$. In view of [M-O-S], $\eta(L)$ is a maximal totally isotropic subspace of \mathbf{F}_4^n , and every vector $x \in \eta(L)$ has at least four nonzero coordinates. Since $Tr \circ h(x, x) \geq 1$, for every x belonging to \mathcal{H}^* , it follows easily that the set of vectors of norm 2 in L_n is nD_4 . Clearly L_n is unimodular.

For $n = 6$, this gives the unique unimodular \mathbf{Z} -lattice of type $6D_4$ which is also an \mathcal{H} -lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable \mathbf{Z} -lattices of type nD_4 for $n \leq 14$, which are \mathcal{H} -lattices. The following construction gives an example of a \mathbf{Z} -lattice of type $8D_4$ which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a \mathbf{Z} -lattice of type nD_4 which is not an \mathcal{H} -lattice is 32). For $1 \leq k \leq 8$, let ρ_k be equal to ξ if k is even and

let ρ_k be equal to 1 if k is odd. Let $\beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i$, $\beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$

for $n \leq j \leq 2$, $\beta_7 = \xi \cdot \sum_{i=1}^4 e_{2i}$ and $\beta_8 = \bar{\xi} \cdot \sum_{i=1}^4 e_{2i-1}$. Let Λ be the \mathbf{Z} -linear subspace of \mathcal{H}^{*8} spanned by \mathcal{H}^8 and $\{\lambda \beta_i\}_{1 \leq i \leq 8}$. Then $\eta(\Lambda)$ is a maximal totally isotropic subspace of $(\mathbf{F}_4^8, Tr \circ \eta(h))$. It can be easily checked that Λ is a \mathbf{Z} -lattice of type $8D_4$. Further $\eta(\Lambda)$ is not invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_8)$ for any choice of α_i in $\{\omega, \omega^2\}$. Thus in view of Proposition 4.7, the lattice Λ does not admit any perfect isometry. The above construction easily generalizes to give a family of \mathbf{Z} -lattices Λ_{4n} of dimension $16m$, $m \geq 2$, which are not \mathcal{H} -lattices.

ACKNOWLEDGEMENT. I thank Eva Bayer for critically going through an earlier version of the manuscript and for making useful comments, which led to a better exposition of this work. My thanks are also due to my teachers R. Parimala and R. Sridharan who showed deep interest in my work. I thank H.G. Quebbemann for carefully going through the manuscript and making valuable suggestion.