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repeating the above argument we obtain a similar decomposition of $N_1: N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_j is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. PROPOSITION. *Let L be a unimodular \mathbf{Z} -lattice of type nD_4 such that $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on \mathcal{H}^{*n} , where δ_i is the isometry on \mathcal{H}^* given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ .*

Proof. Let σ be a perfect isometry of $(L, \text{Tr} \circ h)$. Then σ induces an automorphism of \mathcal{H}^n and extends naturally to a perfect isometry of \mathcal{H}^{*n} . In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of \mathbf{F}_4^n , leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α . Let δ_i denote left multiplication on \mathcal{H}^* by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since δ induces an isometry of \mathcal{H}^{*n} which fixes \mathcal{H}^n and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. THEOREM. *Let (L, S) be an unimodular \mathbf{Z} -lattice of type nD_4 . Then, L has a perfect isometry if and only if there exists an \mathcal{H} -lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every \mathcal{H} -lattice admits a perfect isometry (3.2). Conversely let (L, S) be a \mathbf{Z} -lattice of type nD_4 , which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ and $S = \text{Tr} \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \dots, n\}$ such that L is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$ be defined by $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \text{id}$ for $i \in T$ and f_i is the involution on \mathcal{H} for $i \notin T$. Then it is easy to check that f is an isometry of $(L, \text{Tr} \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$

Clearly L' is invariant under left multiplication by ξ . Further, since $\mathcal{P}L' \subseteq \mathcal{P}\mathcal{H}^{*n} \subseteq \mathcal{H}^n \subseteq L'$, it follows that L' is an \mathcal{H} -lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type D_4 .

5.3. THEOREM. *Let (L, S) , be a positive definite unimodular symmetric bilinear space over \mathbf{Z} , of rank n . Suppose that the set of vectors of norm 2 form a root system of type*

$$R = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

with, $\sum_{1 \leq i \leq p} 2k_i + 6q + 8r + 4s = n$. Then the following hold:

(i) *The \mathbf{Z} -lattice L decomposes as $L = L_1 \perp L_2 \perp L_3$, where each L_i is unimodular, with associated root systems of type $R_1 = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6$,*

$R_2 = rE_8$, $R_3 = sD_4$, respectively.

(ii) *The \mathbf{Z} -lattice L admits a perfect isometry if and only if L_3 is isometric to the trace form of an \mathcal{H} -lattice.*

(iii) *If L admits a perfect isometry, then it admits a perfect isometry σ such that the induced map $\eta(\sigma)$ on $\mathbf{Z}R^\#/\mathbf{Z}R$, corresponds to multiplication by -1 , on the components corresponding to A_{2k_i} , E_6 , and E_8 , and to multiplication by ω , on the components corresponding to D_4 .*

Proof. (i) Since E_8 is unimodular, it is clear that $L = L_2 \perp K$, where $L_2 \simeq r\mathbf{Z}E_8$, and K is unimodular with associated root system of type $R_1 \perp R_3$. So to prove (i), it is enough to prove that K decomposes as $L_1 \perp L_3$. This would follow if we show that $\eta(K)$ decomposes as, $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^\#/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^\#/\mathbf{Z}R_3)$.

Let $z = (x, y) \in \eta(K)$, with x in $\mathbf{Z}R_1^\#/\mathbf{Z}R_1$ and y in $\mathbf{Z}R_3^\#/\mathbf{Z}R_3$. Since $\mathbf{Z}R_1^\#/\mathbf{Z}R_1$ is a group of exponent 3. $\prod_{1 \leq i \leq p} (2k_i + 1)$, and $\mathbf{Z}R_3^\#/\mathbf{Z}R_3 \simeq \mathbf{F}_4^m$,

it follows that, $(0, y) = 3(\prod_{1 \leq i \leq p} (2k_i + 1))z \in \eta(K)$. Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

5.4. Examples. We conclude this section by giving some examples of \mathcal{H} -lattices of type nD_4 as well as \mathbf{Z} -lattices of type nD_4 which are not \mathcal{H} -lattices. Let $\{e_k\}_{1 \leq k \leq n}$ denote the standard \mathcal{H} -basis of \mathcal{H}^n . We

consider two cases. For $n = 4m$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \leq j \leq 2m-2$, and

$$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}. \text{ For } n = 4m + 2, \text{ let } \varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k, 0 \leq j \leq 2m-1,$$

and $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$. Let $\lambda = 1/1 + i$ and let L_n be the \mathcal{H} -lattice generated by $\mathcal{H}^n \cup \{\lambda \varepsilon_1, \lambda \varepsilon_2, \dots, \lambda \varepsilon_{n/2}\}$. In view of [M-O-S], $\eta(L)$ is a maximal totally isotropic subspace of \mathbf{F}_4^n , and every vector $x \in \eta(L)$ has at least four nonzero coordinates. Since $Tr \circ h(x, x) \geq 1$, for every x belonging to \mathcal{H}^* , it follows easily that the set of vectors of norm 2 in L_n is nD_4 . Clearly L_n is unimodular.

For $n = 6$, this gives the unique unimodular \mathbf{Z} -lattice of type $6D_4$ which is also an \mathcal{H} -lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable \mathbf{Z} -lattices of type nD_4 for $n \leq 14$, which are \mathcal{H} -lattices. The following construction gives an example of a \mathbf{Z} -lattice of type $8D_4$ which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a \mathbf{Z} -lattice of type nD_4 which is not an \mathcal{H} -lattice is 32). For $1 \leq k \leq 8$, let ρ_k be equal to ξ if k is even and

$$\text{let } \rho_k \text{ be equal to } 1 \text{ if } k \text{ is odd. Let } \beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i, \beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$$

$$\text{for } n \leq j \leq 2, \beta_7 = \xi \cdot \sum_{i=1}^4 e_{2i} \text{ and } \beta_8 = \bar{\xi} \cdot \sum_{i=1}^4 e_{2i-1}. \text{ Let } \Lambda \text{ be the } \mathbf{Z}\text{-linear}$$

subspace of \mathcal{H}^{*8} spanned by \mathcal{H}^8 and $\{\lambda \beta_i\}_{1 \leq i \leq 8}$. Then $\eta(\Lambda)$ is a maximal totally isotropic subspace of $(\mathbf{F}_4^8, Tr \circ \eta(h))$. It can be easily checked that Λ is a \mathbf{Z} -lattice of type $8D_4$. Further $\eta(\Lambda)$ is not invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_8)$ for any choice of α_i in $\{\omega, \omega^2\}$. Thus in view of Proposition 4.7, the lattice Λ does not admit any perfect isometry. The above construction easily generalizes to give a family of \mathbf{Z} -lattices Λ_{4n} of dimension $16m$, $m \geq 2$, which are not \mathcal{H} -lattices.

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