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repeating the above argument we obtain a similar decomposition of $N_1: N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where each M_j is invariant under diag $(\alpha_1, ..., \alpha_i, ..., \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§ 5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. PROPOSITION. Let L be a unimodular **Z**-lattice of type nD_4 such that $\mathcal{H}^n \subset L \subset \mathcal{H}^{*^n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \operatorname{diag}(\delta_1, ..., \delta_i, ..., \delta_n)$ on \mathcal{H}^{*^n} , where δ_i is the isometry on \mathcal{H}^* given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ .

Proof. Let σ be a perfect isometry of $(L, Tr \circ h)$. Then σ induces an automorphism of \mathcal{H}^n and extends naturally to a perfect isometry of \mathcal{H}^{*n} . In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of \mathbf{F}_4^n , leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \operatorname{diag}(\alpha_1, ..., \alpha_i, ..., \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α . Let δ_i denote left multiplication on \mathcal{H}^* by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \operatorname{diag}(\delta_1, ..., \delta_i, ..., \delta_n)$. Since δ induces an isometry of \mathcal{H}^{*n} which fixes \mathcal{H}^n and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. THEOREM. Let (L, S) be an unimodular **Z**-lattice of type nD_4 . Then, L has a perfect isometry if and only if there exists an \mathcal{H} -lattice (L', S') such that $L \simeq L'$.

Proof. Clearly every \mathcal{H} -lattice admits a perfect isometry (3.2). Conversely let (L, S) be a **Z**-lattice of type nD_4 , which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, ..., n\}$ such that L is invariant under $\delta = (\delta_1, ..., \delta_i, ..., \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by ξ for $i \notin T$. Let $f: \mathcal{H}^n \to \mathcal{H}^n$ be defined by $f = \text{diag}(f_1, ..., f_i, ..., f_n)$ where $f_i = \text{id}$ for $i \in T$ and $f_i = \text{the}$ involution on \mathcal{H} for $i \notin T$. Then it is easy to check that f is an isometry of $(L, Tr \circ h)$ onto (L', S') where, L' = f(L), and,

$$S'(x,y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \in T} (\bar{x}_i y_i + \bar{y}_i x_i) .$$

Clearly L' is invariant under left multiplication by ξ . Further, since $\mathscr{P}L' \subseteq \mathscr{PH}^n \subseteq \mathscr{H}^n \subseteq L'$, it follows that L' is an \mathscr{H} -lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type D_4 .

5.3. THEOREM. Let (L, S), be a positive definite unimodular symmetric bilinear space over \mathbb{Z} , of rank n. Suppose that the set of vectors of norm 2 form a root system of type

$$R = \underset{1 \leq i \leq p}{\perp} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

with,
$$\sum_{1 \le i \le p} 2k_i + 6q + 8r + 4s = n$$
. Then the following hold:

(i) The **Z**-lattice L decomposes as $L = L_1 \perp L_2 \perp L_3$, where each L_i is unimodular, with associated root systems of type $R_1 = \underset{1 \leq i \leq p}{\perp} A_{2k_i} \perp qE_6$,

 $R_2 = rE_8$, $R_3 = sD_4$, respectively.

- (ii) The **Z**-lattice L admits a perfect isometry if and only if L_3 is isometric to the trace form of an \mathcal{H} -lattice.
- (iii) If L admits a perfect isometry, then it admits a perfect isometry σ such that the induced map $\eta(\sigma)$ on $\mathbb{Z}R^{\#}/\mathbb{Z}R$, corresponds to multiplication by -1, on the components corresponding to A_{2k_i} , E_6 , and E_8 , and to multiplication by ω , on the components corresponding to D_4 .

Proof. (i) Since E_8 is unimodular, it is clear that $L = L_2 \perp K$, where $L_2 \simeq r \mathbf{Z} E_8$, and K is unimodular with associated root system of type $R_1 \perp R_3$. So to prove (i), it is enough to prove that K decomposes as $L_1 \perp L_3$. This would follow if we show that $\eta(K)$ decomposes as, $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^\#/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^\#/\mathbf{Z}R_3)$.

Let $z = (x, y) \in \eta(K)$, with x in $\mathbb{Z}R_1^\#/\mathbb{Z}R_1$ and y in $\mathbb{Z}R_3^\#/\mathbb{Z}R_3$. Since $\mathbb{Z}R_1^\#/\mathbb{Z}R_1$ is a group of exponent 3. $\prod_{1 \le i \le p} (2k_i + 1)$, and $\mathbb{Z}R_3^\#/\mathbb{Z}R_3 \simeq \mathbb{F}_4^m$,

it follows that, $(0, y) = 3(\prod_{1 \le i \le p} (2k_i + 1))z \in \eta(K)$. Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

5.4. Examples. We conclude this section by giving some examples of \mathcal{H} -lattices of type nD_4 as well as **Z**-lattices of type nD_4 which are not \mathcal{H} -lattices. Let $\{e_k\}_{1 \leq k \leq n}$ denote the standard \mathcal{H} -basis of \mathcal{H}^n . We

consider two cases. For
$$n = 4m$$
, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \le j \le 2m-2$, and

$$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}.$$
 For $n = 4m+2$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \le j \le 2m-1$,

and $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$. Let $\lambda = 1/1 + i$ and let L_n be the \mathcal{H} -lattice generated by $\mathcal{H}^n \cup \{\lambda \varepsilon_1, \lambda \varepsilon_2, ..., \lambda \varepsilon_{n/2}\}$. In view of [M-O-S], $\eta(L)$ is a maximal totally isotropic subspace of \mathbf{F}_4^n , and every vector $x \in \eta(L)$ has at least four nonzero coordinates. Since $Tr \circ h(x, x) \geqslant 1$, for every x belonging to \mathcal{H}^* , it follows easily that the set of vectors of norm 2 in L_n is nD_4 . Clearly L_n is unimodular.

For n=6, this gives the unique unimodular **Z**-lattice of type $6D_4$ which is also an \mathcal{H} -lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable **Z**-lattices of type nD_4 for $n \le 14$, which are \mathcal{H} -lattices. The following construction gives an example of a **Z**-lattice of type $8D_4$ which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a **Z**-lattice of type nD_4 which is not an \mathcal{H} -lattice is 32). For $1 \le k \le 8$, let ρ_k be equal to ξ if k is even and

let
$$\rho_k$$
 be equal to 1 if k is odd. Let $\beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i$, $\beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$

for
$$n \le j \le 2$$
, $\beta_7 = \xi \cdot \sum_{i=1}^4 e_{2i}$ and $\beta_8 = \bar{\xi} \cdot \sum_{i=1}^4 e_{2i-1}$. Let Λ be the **Z**-linear

subspace of \mathcal{H}^{*8} spanned by \mathcal{H}^{8} and $\{\lambda\beta_{i}\}_{1\leqslant i\leqslant 8}$. Then $\eta(\Lambda)$ is a maximal totally isotropic subspace of $(\mathbf{F}_{4}^{8}, Tr\circ\eta(h))$. It can be easily checked that Λ is a **Z**-lattice of type $8D_{4}$. Further $\eta(\Lambda)$ is not invariant under diag $(\alpha_{1},...,\alpha_{i},...,\alpha_{8})$ for any choice of α_{i} in $\{\omega,\omega^{2}\}$. Thus in view of Proposition 4.7, the lattice Λ does not admit any perfect isometry. The above construction easily generalizes to give a family of **Z**-lattices Λ_{4n} of dimension 16m, $m \geqslant 2$, which are not \mathcal{H} -lattices.

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