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**Autor:** Shastri, Parvati
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repeating the above argument we obtain a similar decomposition of $N_1 : N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$, where each $M_j$ is invariant under $\operatorname{diag}(\alpha_1, \ldots, \alpha_i, \ldots \alpha_n)$, $\alpha_i$ being in $\{\omega, \omega^2\}$.

## §5. Main Theorem and examples

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. Proposition. *Let $L$ be a unimodular $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ such that $\mathscr{H}^n \subset L \subset \mathscr{H}^{*n}$. If $L$ admits a perfect isometry, then there exists an isometry $\delta = \operatorname{diag}(\delta_1, \ldots, \delta_i, \ldots, \delta_n)$ on $\mathscr{H}^{*n}$, where $\delta_i$ is the isometry on $\mathscr{H}^*$ given by left multiplication by $\xi$ or right multiplication by $\bar{\xi}$ such that $L$ is invariant under $\delta$.*

*Proof.* Let $\sigma$ be a perfect isometry of $(L, Tr \circ h)$. Then $\sigma$ induces an automorphism of $\mathscr{H}^n$ and extends naturally to a perfect isometry of $\mathscr{H}^{*n}$. In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of $\mathbf{F}_4^n$, leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ with $\alpha_i$ in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under $\alpha$. Let $\delta_i$ denote left multiplication on $\mathscr{H}^*$ by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \operatorname{diag}(\delta_1, \ldots, \delta_i, \ldots, \delta_n)$. Since $\delta$ induces an isometry of $\mathscr{H}^{*n}$ which fixes $\mathscr{H}^n$ and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that $\delta$ leaves $L$ invariant.

5.2. Theorem. *Let $(L, S)$ be an unimodular $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$. Then, $L$ has a perfect isometry if and only if there exists an $\mathscr{H}$-lattice $(L', S')$ such that $L \simeq L'$.*

*Proof.* Clearly every $\mathscr{H}$-lattice admits a perfect isometry (3.2). Conversely let $(L, S)$ be a $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$, which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset $T$ of $\{1, 2, \ldots, n\}$ such that $L$ is invariant under $\delta = (\delta_1, \ldots, \delta_i, \ldots, \delta_n)$, where $\delta_i$ is left multiplication by $\xi$ for $i \in T$ and $\delta_i$ is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f : \mathscr{H}^n \to \mathscr{H}^n$ be defined by $f = \operatorname{diag}(f_1, \ldots, f_i, \ldots, f_n)$ where $f_i = \mathrm{id}$ for $i \in T$ and $f_i =$ the involution on $\mathscr{H}$ for $i \notin T$. Then it is easy to check that $f$ is an isometry of $(L, Tr \circ h)$ onto $(L', S')$ where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i) .$$

Clearly $L'$ is invariant under left multiplication by $\xi$. Further, since $\mathscr{P}L' \subseteq \mathscr{P}\mathscr{H}^{*n} \subseteq \mathscr{H}^n \subseteq L'$, it follows that $L'$ is an $\mathscr{H}$-lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type $D_4$.

5.3. THEOREM. *Let* $(L, S)$, *be a positive definite unimodular symmetric bilinear space over* $\mathbf{Z}$, *of rank* $n$. *Suppose that the set of vectors of norm 2 form a root system of type*

$$R = \perp_{1 \leqslant i \leqslant p} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

*with,* $\sum_{1 \leqslant i \leqslant p} 2k_i + 6q + 8r + 4s = n$. *Then the following hold:*

(i) *The* $\mathbf{Z}$*-lattice* $L$ *decomposes as* $L = L_1 \perp L_2 \perp L_3$, *where each* $L_i$ *is unimodular, with asociated root systems of type* $R_1 = \perp_{1 \leqslant i \leqslant p} A_{2k_i} \perp qE_6$, $R_2 = rE_8$, $R_3 = sD_4$, *respectively.*

(ii) *The* $\mathbf{Z}$*-lattice* $L$ *admits a perfect isometry if and only if* $L_3$ *is isometric to the trace form of an* $\mathscr{H}$*-lattice.*

(iii) *If* $L$ *admits a perfect isometry, then it admits a perfect isometry* $\sigma$ *such that the induced map* $\eta(\sigma)$ *on* $\mathbf{Z}R^{\#}/\mathbf{Z}R$, *corresponds to multiplication by* $-1$, *on the components corresponding to* $A_{2k_i}$, $E_6$, *and* $E_8$, *and to multiplication by* $\omega$, *on the components corresponding to* $D_4$.

*Proof.* (i) Since $E_8$ is unimodular, it is clear that $L = L_2 \perp K$, where $L_2 \simeq r\mathbf{Z}E_8$, and $K$ is unimodular with associated root system of type $R_1 \perp R_3$. So to prove (i), it is enough to prove that $K$ decomposes as $L_1 \perp L_3$. This would follow if we show that $\eta(K)$ decomposes as, $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3)$.

Let $z = (x, y) \in \eta(K)$, with $x$ in $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ and $y$ in $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3$. Since $\mathbf{Z}R_1^{\#}/\mathbf{Z}R_1$ is a group of exponent $3 \cdot \prod_{1 \leqslant i \leqslant p} (2k_i + 1)$, and $\mathbf{Z}R_3^{\#}/\mathbf{Z}R_3 \simeq \mathbf{F}_4^m$, it follows that, $(0, y) = 3(\prod_{1 \leqslant i \leqslant p} (2k_i + 1))z \in \eta(K)$. Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

5.4. *Examples.* We conclude this section by giving some examples of $\mathscr{H}$-lattices of type $nD_4$ as well as $\mathbf{Z}$-lattices of type $nD_4$ which are not $\mathscr{H}$-lattices. Let $\{e_k\}_{1 \leqslant k \leqslant n}$ denote the standard $\mathscr{H}$-basis of $\mathscr{H}^n$. We consider two cases. For $n = 4m$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$, $0 \leqslant j \leqslant 2m - 2$, and

$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}$. For $n = 4m + 2$, let $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k, 0 \leqslant j \leqslant 2m - 1$, and $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$. Let $\lambda = 1/1 + i$ and let $L_n$ be the $\mathscr{H}$-lattice generated by $\mathscr{H}^n \cup \{\lambda\varepsilon_1, \lambda\varepsilon_2, \ldots, \lambda\varepsilon_{n/2}\}$. In view of [M-O-S], $\eta(L)$ is a maximal totally isotropic subspace of $\mathbf{F}_4^n$, and every vector $x \in \eta(L)$ has at least four nonzero coordinates. Since $Tr \circ h(x, x) \geqslant 1$, for every $x$ belonging to $\mathscr{H}^*$, it follows easily that the set of vectors of norm 2 in $L_n$ is $n\mathrm{D}_4$. Clearly $L_n$ is unimodular.

For $n = 6$, this gives the unique unimodular $\mathbf{Z}$-lattice of type $6\mathrm{D}_4$ which is also an $\mathscr{H}$-lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable $\mathbf{Z}$-lattices of type $n\mathrm{D}_4$ for $n \leqslant 14$, which are $\mathscr{H}$-lattices. The following construction gives an example of a $\mathbf{Z}$-lattice of type $8\mathrm{D}_4$ which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ which is not an $\mathscr{H}$-lattice is 32). For $1 \leqslant k \leqslant 8$, let $\rho_k$ be equal to $\xi$ if $k$ is even and let $\rho_k$ be equal to 1 if $k$ is odd. Let $\beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i$, $\beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$ for $n \leqslant j \leqslant 2$, $\beta_7 = \xi \cdot \sum_{i=1}^{4} e_{2i}$ and $\beta_8 = \bar{\xi} \cdot \sum_{i=1}^{4} e_{2i-1}$. Let $\Lambda$ be the $\mathbf{Z}$-linear subspace of $\mathscr{H}^{*8}$ spanned by $\mathscr{H}^8$ and $\{\lambda\beta_i\}_{1 \leqslant i \leqslant 8}$. Then $\eta(\Lambda)$ is a maximal totally isotropic subspace of $(\mathbf{F}_4^8, Tr \circ \eta(h))$. It can be easily checked that $\Lambda$ is a $\mathbf{Z}$-lattice of type $8\mathrm{D}_4$. Further $\eta(\Lambda)$ is not invariant under $\mathrm{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_8)$ for any choice of $\alpha_i$ in $\{\omega, \omega^2\}$. Thus in view of Proposition 4.7, the lattice $\Lambda$ does not admit any perfect isometry. The above construction easily generalizes to give a family of $\mathbf{Z}$-lattices $\Lambda_{4n}$ of dimension $16m$, $m \geqslant 2$, which are not $\mathscr{H}$-lattices.

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