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repeating the above argument we obtain a similar decomposition of  $N_1: N_1 = M_2 \oplus N_2$ . This process terminates in a finite number of steps and we obtain a decomposition  $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$ , where each  $M_j$  is invariant under  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ ,  $\alpha_i$  being in  $\{\omega, \omega^2\}$ .

## §5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

**5.1. PROPOSITION.** *Let  $L$  be a unimodular  $\mathbf{Z}$ -lattice of type  $nD_4$  such that  $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$ . If  $L$  admits a perfect isometry, then there exists an isometry  $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$  on  $\mathcal{H}^{*n}$ , where  $\delta_i$  is the isometry on  $\mathcal{H}^*$  given by left multiplication by  $\xi$  or right multiplication by  $\bar{\xi}$  such that  $L$  is invariant under  $\delta$ .*

*Proof.* Let  $\sigma$  be a perfect isometry of  $(L, \text{Tr} \circ h)$ . Then  $\sigma$  induces an automorphism of  $\mathcal{H}^n$  and extends naturally to a perfect isometry of  $\mathcal{H}^{*n}$ . In view of ([K], p. 179),  $\eta(\sigma)$  is a perfect isomorphism of  $\mathbf{F}_4^n$ , leaving  $\eta(L)$  invariant. Therefore by Proposition 4.7 there exists  $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$  with  $\alpha_i$  in  $\{\omega, \omega^2\}$  such that  $\eta(L)$  is invariant under  $\alpha$ . Let  $\delta_i$  denote left multiplication on  $\mathcal{H}^*$  by  $\xi = (1 + i + j + k)/2$  if  $\alpha_i = \omega$  and right multiplication by  $\bar{\xi} = (1 - i - j - k)/2$ , if  $\alpha_i = \omega^2$ . Let  $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ . Since  $\delta$  induces an isometry of  $\mathcal{H}^{*n}$  which fixes  $\mathcal{H}^n$  and  $\eta(\delta) = \alpha$  leaves  $\eta(L)$  invariant it follows that  $\delta$  leaves  $L$  invariant.

**5.2. THEOREM.** *Let  $(L, S)$  be an unimodular  $\mathbf{Z}$ -lattice of type  $nD_4$ . Then,  $L$  has a perfect isometry if and only if there exists an  $\mathcal{H}$ -lattice  $(L', S')$  such that  $L \simeq L'$ .*

*Proof.* Clearly every  $\mathcal{H}$ -lattice admits a perfect isometry (3.2). Conversely let  $(L, S)$  be a  $\mathbf{Z}$ -lattice of type  $nD_4$ , which admits a perfect isometry. In view of Proposition 2.1, we can assume that  $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$  and  $S = \text{Tr} \circ h$ . By Proposition 4.7 there exists a subset  $T$  of  $\{1, 2, \dots, n\}$  such that  $L$  is invariant under  $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$ , where  $\delta_i$  is left multiplication by  $\xi$  for  $i \in T$  and  $\delta_i$  is right multiplication by  $\bar{\xi}$  for  $i \notin T$ . Let  $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$  be defined by  $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$  where  $f_i = \text{id}$  for  $i \in T$  and  $f_i$  is the involution on  $\mathcal{H}$  for  $i \notin T$ . Then it is easy to check that  $f$  is an isometry of  $(L, \text{Tr} \circ h)$  onto  $(L', S')$  where,  $L' = f(L)$ , and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$

Clearly  $L'$  is invariant under left multiplication by  $\xi$ . Further, since  $\mathcal{P}L' \subseteq \mathcal{P}\mathcal{H}^{*n} \subseteq \mathcal{H}^n \subseteq L'$ , it follows that  $L'$  is an  $\mathcal{H}$ -lattice.

Finally, we have the following analogue of Proposition 1.5 for the case of lattices having components of type  $D_4$ .

**5.3. THEOREM.** *Let  $(L, S)$ , be a positive definite unimodular symmetric bilinear space over  $\mathbf{Z}$ , of rank  $n$ . Suppose that the set of vectors of norm 2 form a root system of type*

$$R = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6 \perp rE_8 \perp sD_4$$

with,  $\sum_{1 \leq i \leq p} 2k_i + 6q + 8r + 4s = n$ . Then the following hold:

(i) *The  $\mathbf{Z}$ -lattice  $L$  decomposes as  $L = L_1 \perp L_2 \perp L_3$ , where each  $L_i$  is unimodular, with associated root systems of type  $R_1 = \bigoplus_{1 \leq i \leq p} A_{2k_i} \perp qE_6$ ,*

$R_2 = rE_8$ ,  $R_3 = sD_4$ , respectively.

(ii) *The  $\mathbf{Z}$ -lattice  $L$  admits a perfect isometry if and only if  $L_3$  is isometric to the trace form of an  $\mathcal{H}$ -lattice.*

(iii) *If  $L$  admits a perfect isometry, then it admits a perfect isometry  $\sigma$  such that the induced map  $\eta(\sigma)$  on  $\mathbf{Z}R^\#/\mathbf{Z}R$ , corresponds to multiplication by  $-1$ , on the components corresponding to  $A_{2k_i}$ ,  $E_6$ , and  $E_8$ , and to multiplication by  $\omega$ , on the components corresponding to  $D_4$ .*

*Proof.* (i) Since  $E_8$  is unimodular, it is clear that  $L = L_2 \perp K$ , where  $L_2 \simeq r\mathbf{Z}E_8$ , and  $K$  is unimodular with associated root system of type  $R_1 \perp R_3$ . So to prove (i), it is enough to prove that  $K$  decomposes as  $L_1 \perp L_3$ . This would follow if we show that  $\eta(K)$  decomposes as,  $\eta(K) = \eta(K) \cap (\mathbf{Z}R_1^\#/\mathbf{Z}R_1) \perp \eta(K) \cap (\mathbf{Z}R_3^\#/\mathbf{Z}R_3)$ .

Let  $z = (x, y) \in \eta(K)$ , with  $x$  in  $\mathbf{Z}R_1^\#/\mathbf{Z}R_1$  and  $y$  in  $\mathbf{Z}R_3^\#/\mathbf{Z}R_3$ . Since  $\mathbf{Z}R_1^\#/\mathbf{Z}R_1$  is a group of exponent 3.  $\prod_{1 \leq i \leq p} (2k_i + 1)$ , and  $\mathbf{Z}R_3^\#/\mathbf{Z}R_3 \simeq \mathbf{F}_4^m$ ,

it follows that,  $(0, y) = 3(\prod_{1 \leq i \leq p} (2k_i + 1))z \in \eta(K)$ . Hence (i) follows.

The results (ii) and (iii) follow from (i), (5.2) and ([K], Prop. 4).

**5.4. Examples.** We conclude this section by giving some examples of  $\mathcal{H}$ -lattices of type  $nD_4$  as well as  $\mathbf{Z}$ -lattices of type  $nD_4$  which are not  $\mathcal{H}$ -lattices. Let  $\{e_k\}_{1 \leq k \leq n}$  denote the standard  $\mathcal{H}$ -basis of  $\mathcal{H}^n$ . We

consider two cases. For  $n = 4m$ , let  $\varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k$ ,  $0 \leq j \leq 2m-2$ , and

$$\varepsilon_{2m} = \sum_{k=0}^{2m-1} e_{2k+1}. \text{ For } n = 4m + 2, \text{ let } \varepsilon_{j+1} = \sum_{k=2j+1}^{2j+4} e_k, 0 \leq j \leq 2m-1,$$

and  $\varepsilon_{2m+1} = \sum_{k=0}^{2m-1} e_{2k+1} + \xi e_{4m+1} + \bar{\xi} e_{4m}$ . Let  $\lambda = 1/1 + i$  and let  $L_n$  be the  $\mathcal{H}$ -lattice generated by  $\mathcal{H}^n \cup \{\lambda \varepsilon_1, \lambda \varepsilon_2, \dots, \lambda \varepsilon_{n/2}\}$ . In view of [M-O-S],  $\eta(L)$  is a maximal totally isotropic subspace of  $\mathbf{F}_4^n$ , and every vector  $x \in \eta(L)$  has at least four nonzero coordinates. Since  $Tr \circ h(x, x) \geq 1$ , for every  $x$  belonging to  $\mathcal{H}^*$ , it follows easily that the set of vectors of norm 2 in  $L_n$  is  $nD_4$ . Clearly  $L_n$  is unimodular.

For  $n = 6$ , this gives the unique unimodular  $\mathbf{Z}$ -lattice of type  $6D_4$  which is also an  $\mathcal{H}$ -lattice. In view of [M-O-S], table III, and Proposition 2.3, one can determine all indecomposable  $\mathbf{Z}$ -lattices of type  $nD_4$  for  $n \leq 14$ , which are  $\mathcal{H}$ -lattices. The following construction gives an example of a  $\mathbf{Z}$ -lattice of type  $8D_4$  which does not admit a perfect isometry. (In particular this shows that the smallest dimension for which there exists a  $\mathbf{Z}$ -lattice of type  $nD_4$  which is not an  $\mathcal{H}$ -lattice is 32). For  $1 \leq k \leq 8$ , let  $\rho_k$  be equal to  $\xi$  if  $k$  is even and

$$\text{let } \rho_k \text{ be equal to } 1 \text{ if } k \text{ is odd. Let } \beta_{j+1} = \sum_{i=2j+1}^{2j+4} \rho_i e_i, \beta_{j+4} = \sum_{i=2j+1}^{2j+4} \rho_{i+1} e_i$$

$$\text{for } n \leq j \leq 2, \beta_7 = \xi \cdot \sum_{i=1}^4 e_{2i} \text{ and } \beta_8 = \bar{\xi} \cdot \sum_{i=1}^4 e_{2i-1}. \text{ Let } \Lambda \text{ be the } \mathbf{Z}\text{-linear}$$

subspace of  $\mathcal{H}^{*8}$  spanned by  $\mathcal{H}^8$  and  $\{\lambda \beta_i\}_{1 \leq i \leq 8}$ . Then  $\eta(\Lambda)$  is a maximal totally isotropic subspace of  $(\mathbf{F}_4^8, Tr \circ \eta(h))$ . It can be easily checked that  $\Lambda$  is a  $\mathbf{Z}$ -lattice of type  $8D_4$ . Further  $\eta(\Lambda)$  is not invariant under  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_8)$  for any choice of  $\alpha_i$  in  $\{\omega, \omega^2\}$ . Thus in view of Proposition 4.7, the lattice  $\Lambda$  does not admit any perfect isometry. The above construction easily generalizes to give a family of  $\mathbf{Z}$ -lattices  $\Lambda_{4n}$  of dimension  $16m$ ,  $m \geq 2$ , which are not  $\mathcal{H}$ -lattices.

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