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Autor: Shastri, Parvati
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Proof. Note that every perfect isometry σ of $\mathscr{H}$ extends naturally to a perfect isometry of $\mathscr{H}^*$, inducing a perfect $\mathbf{F}_2$-isomorphism $\eta(\sigma)$ of $\mathscr{H}^*/\mathscr{H}$, η denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it corresponds to multiplication by ω, where ω denotes a primitive element of $\mathbf{F}_4$ over $\mathbf{F}_2$.*

Proof. An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of $\mathbf{F}_4$, corresponds to multiplication by ω, ω being as above.

3.6. PROPOSITION. *Let L be a $\mathbf{Z}$-lattice such that $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$. If L is an $\mathscr{H}$-lattice, then L has a perfect isometry, which corresponds to multiplication by ω, on the quotient $\mathscr{H}^{*n}/\mathscr{H}^n$.*

Proof. Multiplication by ξ is a perfect isometry of $\mathscr{H}^n$ which extends naturally to a perfect isometry of $\mathscr{H}^{*n}$. Clearly the induced map on the quotient $\mathscr{H}^{*n}/\mathscr{H}^n$ is multiplication by ω. Since L is an $\mathscr{H}$-module, it preserves L as well.

In particular,

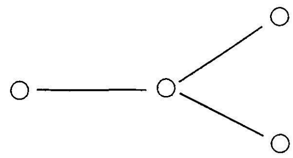
3.7. COROLLARY. *Every $\mathscr{H}$-lattice $(L, Tr \circ h)$ of type $n\mathrm{D}_4$ has a perfect isometry.*

It is but natural to ask whether every $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ which has a perfect isometry necessarily admits the structure of an $\mathscr{H}$-lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system $n\mathrm{D}_4$.

§4. AUTOMORPHISMS OF THE ROOT SYSTEM $n\mathrm{D}_4$ AND PERFECT ISOMETRIES

For any root system R, let $\mathscr{W}(\mathrm{R})$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathscr{W}(\mathrm{R})$ is a normal subgroup of $Aut\,\mathrm{R}$, which preserves every $\mathbf{Z}$-lattice L such that $\mathbf{Z}\mathrm{R} \subseteq L \subseteq \mathbf{Z}\mathrm{R}^{\#}$. We thus get a natural map $\eta : Aut\,\mathrm{R}/\mathscr{W}(\mathrm{R}) \to Aut_{\mathbf{Z}}(\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R})$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.

An element σ in $Aut(\mathrm{R})/\mathscr{W}(\mathrm{R})$ preserves L if and only if $\eta(\sigma)$ preserves the corresponding subgroup $\eta(L)$ of $\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R}$. If $\mathrm{R} = \mathrm{D}_4$, $Aut\,\mathrm{R} = \mathscr{W}(\mathrm{R}) \underset{s}{\ltimes} S_3$, where, $\underset{s}{\ltimes}$ denotes the semi direct product and S_3 is the automorphism group of the associated Dynkin diagram:



Consequently, for $\mathrm{R} = n\mathrm{D}_4$, $Aut\,\mathrm{R}/\mathscr{W}(\mathrm{R}) \simeq S_3^n \underset{s}{\ltimes} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\ltimes} S_n$. Thus the elements of $Aut\mathrm{R}/\mathscr{W}(\mathrm{R})$ are "monomial matrices" where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. It acts naturally on $(\mathbf{Z}\mathrm{D}_4^{\#})^n/\mathbf{Z}\mathrm{D}_4^n$. In view of the identification of $\mathbf{Z}\mathrm{D}_4^{\#}/\mathbf{Z}\mathrm{D}_4 \simeq \mathscr{H}^*/\mathscr{H}$, we have the following proposition.

4.1. PROPOSITION.

(a) $Aut(\mathscr{H}^n)/\mathscr{W}(\mathscr{H}^n) \simeq S_3^n \underset{s}{\ltimes} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\ltimes} S_n$.

(b) *If U denotes the group of units of $\mathscr{H}$, then U is a subgroup of $Aut\,\mathscr{H}$ and $U/(\mathscr{W}(\mathscr{H}) \cap U) \simeq \{1, \omega, \omega^2\}$, where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.*

(c) *The conjugation in $\mathscr{H}$ belongs to the Weyl group $\mathscr{W}(\mathscr{H})$.*

Proof. (a) This statement is an immediate consequence of the identification $\mathbf{Z}\mathrm{D}_4 \simeq \mathscr{H}$.

(b) By (a), $Aut\,\mathscr{H}/\mathscr{W}(\mathscr{H}) \simeq S_3 \simeq GL_2(\mathbf{F}_2)$. Since $\eta(U) = \{1, \omega, \omega^2\}$, (b) follows.

(c) The conjugation in $\mathscr{H}$ is a product of reflections defined by i, j and k.

We now consider the perfect isomorphisms of $(\mathscr{H}^{*n})/\mathscr{H}^n$ arising out of $Aut(\mathscr{H}^n)/\mathscr{W}(\mathscr{H}^n)$. We begin by fixing the following notation:

Let $V = \mathbf{F}_4^n = X_1 \perp X_2 \perp \ldots X_n$ with respect to the standard hermitian form on V, where $X_i \simeq \mathbf{F}_4 = \mathbf{F}_2 \oplus \mathbf{F}_2 = \{0, 1, \omega, \omega^2\}$. Let G denote the group of all $n \times n$ monomial matrices with entries in $M_2(\mathbf{F}_2)$, where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. Note that every element of G can be uniquely expressed as $\alpha . \tau$, where α is the diagonal matrix $\mathrm{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$, with α_i in $GL_2(\mathbf{F}_2)$ and τ is an $n \times n$ permutation matrix. We have, 4.2. LEMMA. *Let* σ *belonging to* G *be perfect and let* $X = X_i$ *for some* i. *Let* m *be the smallest positive integer for which* σ^m *maps* X *onto itself. Then* σ^m/X *is perfect.*

Proof. The idea of the proof is similar to ([K], Prop. 2). We show that $(1-\sigma^m)/X$ is surjective. Let $M = \sum_{0 \leqslant i \leqslant m-1} \sigma^i(X)$. Then σ leaves M invariant. Therefore σ is a perfect isomorphism of M. Hence $(1-\sigma)/M$: $M \to M$ is surjective. Let x be an element of X. Since, $(x, 0, \ldots, 0)$ belongs to M, there exists an element y in M such that $(1-\sigma)(y) = (x, 0, \ldots, 0)$. Let $y = (y_0, y_1, \ldots, y_{m-1})$, where y_i belongs to $\sigma^i(X)$. Then,

$$(1-\sigma)(y) = (y_0 - \sigma(y_{m-1}) , \quad y_1 - \sigma(y_0) , \ldots, y_{m-1} - \sigma(y_{m-2})) .$$

Hence, $y_0 - \sigma(y_{m-1}) = x, y_1 = \sigma(y_0), \ldots, y_{m-1} = \sigma(y_{m-2})$. Further, $\sigma(y_{m-1}) = \sigma^2(y_{m-2}) = \ldots = \sigma^m(y_0)$. Thus $(1-\sigma^m)(y_0) = x$. This implies that $(1-\sigma^m)/X$ is surjective.

4.3. COROLLARY. *Let* σ *be an element of* G *which is perfect. Suppose that* $\sigma = \alpha . \tau$, *where* $\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$, $\alpha_i \in GL_2(\mathbf{F}_2)$, $\tau = \tau_1 . \tau_2 \ldots \tau_r$, *and* τ_i *are disjoint cyclic permutations of length* n_i. *Let* T_i *denote the set of indices belonging to the permutation* τ_i. *Then* $(\sigma)^{n_i}/X_j$ *is perfect for every* j *belonging to* T_i.

Proof. Note that for every j belonging to T_i, n_i is the smallest positive integer such that $(\sigma)^{n_i}$ maps X_j onto itself.

4.4. COROLLARY. If σ *is as above, then* $(\sigma)^{n_i}/X_j$ *corresponds to multiplication by* ω *or* ω^2, *for every* j *belonging to* T_i.

Proof. Follows from Corollary 4.3, and Lemma 3.5.

4.5. COROLLARY. *If* σ *is as above, and* $X^{(i)} = \sum_{j \in T_i} X_j$, *then* $(\sigma)^{n_i}/X^{(i)}$ *is the matrix* $\mathrm{diag}(\alpha_1, \ldots, \alpha_j, \ldots \alpha_{n_i})$, *where* α_j *belongs to* $\{\omega, \omega^2\}$.

Proof. Clear from Corollary 4.4.

4.6. PROPOSITION. *Let* σ *be an element of* G *which is perfect and let* $\sigma = \alpha . \tau$, *where* α *and* τ *are as in Corollary 4.4. Then there exists an integer* $l \geqslant 1$, *such that* σ^l *is perfect and* $\sigma^l = \beta . \tau'$, *where* β *is the matrix* $\mathrm{diag}(\beta_1, \ldots, \beta_j, \ldots, \beta_n)$, *with* β_j *in* $GL_2(\mathbf{F}_2)$ *and* τ' *is a product of disjoint cyclic permutations* τ_i *of length* 3^{k_i}.

Proof. Let $\tau = \tau_1 . \tau_2 \dots \tau_r$, where τ_i are disjoint cyclic permutations of length $n_i = 3^{k_i} . l_i$, with $(3, l_i) = 1$. Let l denote the least common multiple of the l_i. We show that σ^l is perfect. By Corollary 4.5, σ^{n_i}/X_j is multiplication by ω or ω^2 for every j belonging to T_i. This implies that $(\sigma)^{n_i l/l_i}/X_j$ corresponds to multiplication by ω or ω^2 for every such j, since $(l/l_i, 3) = 1$ and ω is an element of order 3. Hence, $(\sigma^l)^{3^{k_i}}/X^{(i)}$ is the matrix $\operatorname{diag}(\alpha_1, \dots, \alpha_j, \dots \alpha_{n_i})$ where α_j belongs to $\{\omega, \omega^2\}$. Clearly this implies that $\sigma^l/X^{(i)}$ has no nontrivial fixed point. Since T_i are disjoint, it follows that σ^l has no nontrivial fixed point and hence σ^l is perfect. Obviously σ^l has the required property and the proposition follows.

Now, let M be an $\mathbf{F}_2$-linear subspace of V, which is invariant under a perfect isomorphism σ belonging to G. By the previous proposition, we can assume, by replacing σ by σ^m, that M is invariant under $\sigma = \alpha . \tau$, where α is as in Corollary 4.4 and $\tau = \tau_1 . \tau_2 \dots \tau_r$, τ_i being cyclic permutations of length 3^{k_i}.

4.7. PROPOSITION. *If M is an $\mathbf{F}_2$-linear subspace of V which has a perfect isomorphism σ belonging to G, then M is invariant under the action of a diagonal matrix,* $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ *where each* α_i *belongs to* $\{\omega, \omega^2\}$.

Proof. By replacing σ by a suitable power we may assume that

$$\sigma = \operatorname{diag}(\beta_1, \dots, \beta_i, \dots, \beta_n)\, \tau_1 \tau_2 \dots \tau_r$$

where β_i belongs to $GL_2(\mathbf{F}_2)$ for every i and τ_i are disjoint cyclic permutations of length 3^{k_i}. Further, since disjoint cycles commute we may assume that the length of τ_i is 3^k for $1 \leqslant i \leqslant s$ and the length of τ_i is less than 3^k for $s < i \leqslant r$. Let $T = \{i \in \{1, 2, \dots, n\} \mid i$ occurs in the permutation $\tau_1 \tau_2 \dots \tau_s\}$. Let $M_1 = M \cap \sum_{i \in T} X_i$ and $N_1 = M \cap \sum_{i \notin T} X_i$. We claim that $M = M_1 \oplus N_1$ and that M_1 is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, where each α_i belongs to $\{\omega, \omega^2\}$. Let $(x, y) \in M$, where $x \in \perp_{i \in T} X_i$, $y \in \perp_{i \notin T} X_i$. Since

$$\sigma^{3^k} = \operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)\,,$$

where α_i belongs to $\{\omega, \omega^2\}$ for $i \in T$ and $\alpha_i = 1$ for $i \notin T$, it follows that, $(x, y) + \sigma^{3^k}(x, y) + (\sigma^{3^k})^2(x, y) = (0, y)$ belongs to M. Hence $(x, 0)$ belongs to M as well. Thus $M = M_1 \oplus N_1$. Clearly M_1 is invariant under $\operatorname{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$. Since σ/N_1 is perfect, by repeating the above argument we obtain a similar decomposition of $N_1 : N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$, where each M_j is invariant under $\mathrm{diag}(\alpha_1, \ldots, \alpha_i, \ldots \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. Main Theorem and examples

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. Proposition. *Let L be a unimodular* **Z***-lattice of type $n\mathrm{D}_4$ such that $\mathscr{H}^n \subset L \subset \mathscr{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_i, \ldots, \delta_n)$ on $\mathscr{H}^{*n}$, where δ_i is the isometry on $\mathscr{H}^*$ given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ.*

Proof. Let σ be a perfect isometry of $(L, Tr \circ h)$. Then σ induces an automorphism of $\mathscr{H}^n$ and extends naturally to a perfect isometry of $\mathscr{H}^{*n}$. In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of $\mathbf{F}_4^n$, leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α. Let δ_i denote left multiplication on $\mathscr{H}^*$ by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_i, \ldots, \delta_n)$. Since δ induces an isometry of $\mathscr{H}^{*n}$ which fixes $\mathscr{H}^n$ and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. Theorem. *Let (L, S) be an unimodular* **Z***-lattice of type $n\mathrm{D}_4$. Then, L has a perfect isometry if and only if there exists an $\mathscr{H}$-lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every $\mathscr{H}$-lattice admits a perfect isometry (3.2). Conversely let (L, S) be a **Z**-lattice of type $n\mathrm{D}_4$, which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \ldots, n\}$ such that L is invariant under $\delta = (\delta_1, \ldots, \delta_i, \ldots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f : \mathscr{H}^n \to \mathscr{H}^n$ be defined by $f = \mathrm{diag}(f_1, \ldots, f_i, \ldots, f_n)$ where $f_i = \mathrm{id}$ for $i \in T$ and $f_i =$ the involution on $\mathscr{H}$ for $i \notin T$. Then it is easy to check that f is an isometry of $(L, Tr \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i) .$$