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*Proof.* Note that every perfect isometry  $\sigma$  of  $\mathcal{H}$  extends naturally to a perfect isometry of  $\mathcal{H}^*$ , inducing a perfect  $\mathbb{F}_2$ -isomorphism  $\eta(\sigma)$  of  $\mathcal{H}^*/\mathcal{H}$ ,  $\eta$  denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An  $\mathbb{F}_2$ -linear isomorphism of  $\mathbb{F}_4$  is perfect if and only if it corresponds to multiplication by  $\omega$ , where  $\omega$  denotes a primitive element of  $\mathbb{F}_4$  over  $\mathbb{F}_2$ .*

*Proof.* An  $\mathbb{F}_2$ -linear isomorphism of  $\mathbb{F}_4$  is perfect if and only if it has no fixed point other than the trivial element. Since,  $GL_2(\mathbb{F}_2) \simeq S_3$ , it is easy to see that every perfect isomorphism of  $\mathbb{F}_4$ , corresponds to multiplication by  $\omega$ ,  $\omega$  being as above.

3.6. PROPOSITION. *Let  $L$  be a  $\mathbb{Z}$ -lattice such that  $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ . If  $L$  is an  $\mathcal{H}$ -lattice, then  $L$  has a perfect isometry, which corresponds to multiplication by  $\omega$ , on the quotient  $\mathcal{H}^{*n}/\mathcal{H}^n$ .*

*Proof.* Multiplication by  $\xi$  is a perfect isometry of  $\mathcal{H}^n$  which extends naturally to a perfect isometry of  $\mathcal{H}^{*n}$ . Clearly the induced map on the quotient  $\mathcal{H}^{*n}/\mathcal{H}^n$  is multiplication by  $\omega$ . Since  $L$  is an  $\mathcal{H}$ -module, it preserves  $L$  as well.

In particular,

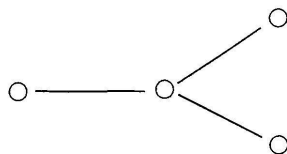
3.7. COROLLARY. *Every  $\mathcal{H}$ -lattice  $(L, Tr \circ h)$  of type  $nD_4$  has a perfect isometry.*

It is but natural to ask whether every  $\mathbb{Z}$ -lattice of type  $nD_4$  which has a perfect isometry necessarily admits the structure of an  $\mathcal{H}$ -lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system  $nD_4$ .

#### §4. AUTOMORPHISMS OF THE ROOT SYSTEM $nD_4$ AND PERFECT ISOMETRIES

For any root system  $R$ , let  $\mathcal{W}(R)$  denote the Weyl group of  $R$  (i.e. the group generated by the reflections defined by the roots). Then  $\mathcal{W}(R)$  is a normal subgroup of  $Aut R$ , which preserves every  $\mathbb{Z}$ -lattice  $L$  such that  $\mathbb{Z}R \subseteq L \subseteq \mathbb{Z}R^\#$ . We thus get a natural map  $\eta: Aut R/\mathcal{W}(R) \rightarrow Aut_{\mathbb{Z}}(\mathbb{Z}R^\#/\mathbb{Z}R)$ . In view of ([H], p. 72; [C-S], p. 432) this is an injection.

An element  $\sigma$  in  $Aut(R)/\mathcal{W}(R)$  preserves  $L$  if and only if  $\eta(\sigma)$  preserves the corresponding subgroup  $\eta(L)$  of  $\mathbb{Z}R^\#/\mathbb{Z}R$ . If  $R = D_4$ ,  $Aut R = \mathcal{W}(R) \rtimes_s S_3$ , where,  $\rtimes_s$  denotes the semi direct product and  $S_3$  is the automorphism group of the associated Dynkin diagram:



Consequently, for  $R = nD_4$ ,  $Aut R/\mathcal{W}(R) \simeq S_3^n \rtimes_s S_n \simeq (GL_2(\mathbb{F}_2))^n \rtimes_s S_n$ . Thus the elements of  $Aut R/\mathcal{W}(R)$  are “monomial matrices” where each row and each column consists of exactly one element of  $GL_2(\mathbb{F}_2)$ . It acts naturally on  $(\mathbb{Z}D_4^\#)^n/\mathbb{Z}D_4^n$ . In view of the identification of  $\mathbb{Z}D_4^\#/\mathbb{Z}D_4 \simeq \mathcal{H}^*/\mathcal{H}$ , we have the following proposition.

#### 4.1. PROPOSITION.

- (a)  $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n) \simeq S_3^n \rtimes_s S_n \simeq (GL_2(\mathbb{F}_2))^n \rtimes_s S_n$ .
- (b) If  $U$  denotes the group of units of  $\mathcal{H}$ , then  $U$  is a subgroup of  $Aut \mathcal{H}$  and  $U/(\mathcal{W}(\mathcal{H}) \cap U) \simeq \{1, \omega, \omega^2\}$ , where  $\mathbb{F}_2(\omega) = \mathbb{F}_4$ .
- (c) The conjugation in  $\mathcal{H}$  belongs to the Weyl group  $\mathcal{W}(\mathcal{H})$ .

*Proof.* (a) This statement is an immediate consequence of the identification  $\mathbb{Z}D_4 \simeq \mathcal{H}$ .

(b) By (a),  $Aut \mathcal{H}/\mathcal{W}(\mathcal{H}) \simeq S_3 \simeq GL_2(\mathbb{F}_2)$ . Since  $\eta(U) = \{1, \omega, \omega^2\}$ , (b) follows.

(c) The conjugation in  $\mathcal{H}$  is a product of reflections defined by  $i, j$  and  $k$ .

We now consider the perfect isomorphisms of  $(\mathcal{H}^{*n})/\mathcal{H}^n$  arising out of  $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n)$ . We begin by fixing the following notation:

Let  $V = \mathbb{F}_4^n = X_1 \perp X_2 \perp \dots \perp X_n$  with respect to the standard hermitian form on  $V$ , where  $X_i \simeq \mathbb{F}_4 = \mathbb{F}_2 \oplus \mathbb{F}_2 = \{0, 1, \omega, \omega^2\}$ . Let  $G$  denote the group of all  $n \times n$  monomial matrices with entries in  $M_2(\mathbb{F}_2)$ , where each row and each column consists of exactly one element of  $GL_2(\mathbb{F}_2)$ . Note that every element of  $G$  can be uniquely expressed as  $\alpha \cdot \tau$ , where  $\alpha$  is the diagonal matrix  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ , with  $\alpha_i$  in  $GL_2(\mathbb{F}_2)$  and  $\tau$  is an  $n \times n$  permutation matrix. We have,

4.2. LEMMA. Let  $\sigma$  belonging to  $G$  be perfect and let  $X = X_i$  for some  $i$ . Let  $m$  be the smallest positive integer for which  $\sigma^m$  maps  $X$  onto itself. Then  $\sigma^m/X$  is perfect.

*Proof.* The idea of the proof is similar to ([K], Prop. 2). We show that  $(1 - \sigma^m)/X$  is surjective. Let  $M = \sum_{0 \leq i \leq m-1} \sigma^i(X)$ . Then  $\sigma$  leaves  $M$  invariant. Therefore  $\sigma$  is a perfect isomorphism of  $M$ . Hence  $(1 - \sigma)/M: M \rightarrow M$  is surjective. Let  $x$  be an element of  $X$ . Since,  $(x, 0, \dots, 0)$  belongs to  $M$ , there exists an element  $y$  in  $M$  such that  $(1 - \sigma)(y) = (x, 0, \dots, 0)$ . Let  $y = (y_0, y_1, \dots, y_{m-1})$ , where  $y_i$  belongs to  $\sigma^i(X)$ . Then,

$$(1 - \sigma)(y) = (y_0 - \sigma(y_{m-1}), y_1 - \sigma(y_0), \dots, y_{m-1} - \sigma(y_{m-2})).$$

Hence,  $y_0 - \sigma(y_{m-1}) = x, y_1 = \sigma(y_0), \dots, y_{m-1} = \sigma(y_{m-2})$ . Further,  $\sigma(y_{m-1}) = \sigma^2(y_{m-2}) = \dots = \sigma^m(y_0)$ . Thus  $(1 - \sigma^m)(y_0) = x$ . This implies that  $(1 - \sigma^m)/X$  is surjective.

4.3. COROLLARY. Let  $\sigma$  be an element of  $G$  which is perfect. Suppose that  $\sigma = \alpha \cdot \tau$ , where  $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ ,  $\alpha_i \in GL_2(\mathbb{F}_2)$ ,  $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$ , and  $\tau_i$  are disjoint cyclic permutations of length  $n_i$ . Let  $T_i$  denote the set of indices belonging to the permutation  $\tau_i$ . Then  $(\sigma)^{n_i}/X_j$  is perfect for every  $j$  belonging to  $T_i$ .

*Proof.* Note that for every  $j$  belonging to  $T_i$ ,  $n_i$  is the smallest positive integer such that  $(\sigma)^{n_i}$  maps  $X_j$  onto itself.

4.4. COROLLARY. If  $\sigma$  is as above, then  $(\sigma)^{n_i}/X_j$  corresponds to multiplication by  $\omega$  or  $\omega^2$ , for every  $j$  belonging to  $T_i$ .

*Proof.* Follows from Corollary 4.3, and Lemma 3.5.

4.5. COROLLARY. If  $\sigma$  is as above, and  $X^{(i)} = \sum_{j \in T_i} X_j$ , then  $(\sigma)^{n_i}/X^{(i)}$  is the matrix  $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$ , where  $\alpha_j$  belongs to  $\{\omega, \omega^2\}$ .

*Proof.* Clear from Corollary 4.4.

4.6. PROPOSITION. Let  $\sigma$  be an element of  $G$  which is perfect and let  $\sigma = \alpha \cdot \tau$ , where  $\alpha$  and  $\tau$  are as in Corollary 4.4. Then there exists an integer  $l \geq 1$ , such that  $\sigma^l$  is perfect and  $\sigma^l = \beta \cdot \tau'$ , where  $\beta$  is the matrix  $\text{diag}(\beta_1, \dots, \beta_j, \dots, \beta_n)$ , with  $\beta_j$  in  $GL_2(\mathbb{F}_2)$  and  $\tau'$  is a product of disjoint cyclic permutations  $\tau_i$  of length  $3^{k_i}$ .

*Proof.* Let  $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$ , where  $\tau_i$  are disjoint cyclic permutations of length  $n_i = 3^{k_i} \cdot l_i$ , with  $(3, l_i) = 1$ . Let  $l$  denote the least common multiple of the  $l_i$ . We show that  $\sigma^l$  is perfect. By Corollary 4.5,  $\sigma^{n_i}/X_j$  is multiplication by  $\omega$  or  $\omega^2$  for every  $j$  belonging to  $T_i$ . This implies that  $(\sigma)^{n_i l/l_i}/X_j$  corresponds to multiplication by  $\omega$  or  $\omega^2$  for every such  $j$ , since  $(l/l_i, 3) = 1$  and  $\omega$  is an element of order 3. Hence,  $(\sigma^l)^{3^{k_i}}/X^{(i)}$  is the matrix  $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$  where  $\alpha_j$  belongs to  $\{\omega, \omega^2\}$ . Clearly this implies that  $\sigma^l/X^{(i)}$  has no nontrivial fixed point. Since  $T_i$  are disjoint, it follows that  $\sigma^l$  has no nontrivial fixed point and hence  $\sigma^l$  is perfect. Obviously  $\sigma^l$  has the required property and the proposition follows.

Now, let  $M$  be an  $\mathbb{F}_2$ -linear subspace of  $V$ , which is invariant under a perfect isomorphism  $\sigma$  belonging to  $G$ . By the previous proposition, we can assume, by replacing  $\sigma$  by  $\sigma^m$ , that  $M$  is invariant under  $\sigma = \alpha \cdot \tau$ , where  $\alpha$  is as in Corollary 4.4 and  $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$ ,  $\tau_i$  being cyclic permutations of length  $3^{k_i}$ .

4.7. PROPOSITION. *If  $M$  is an  $\mathbb{F}_2$ -linear subspace of  $V$  which has a perfect isomorphism  $\sigma$  belonging to  $G$ , then  $M$  is invariant under the action of a diagonal matrix,  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$  where each  $\alpha_i$  belongs to  $\{\omega, \omega^2\}$ .*

*Proof.* By replacing  $\sigma$  by a suitable power we may assume that

$$\sigma = \text{diag}(\beta_1, \dots, \beta_i, \dots, \beta_n) \tau_1 \tau_2 \dots \tau_r$$

where  $\beta_i$  belongs to  $GL_2(\mathbb{F}_2)$  for every  $i$  and  $\tau_i$  are disjoint cyclic permutations of length  $3^{k_i}$ . Further, since disjoint cycles commute we may assume that the length of  $\tau_i$  is  $3^k$  for  $1 \leq i \leq s$  and the length of  $\tau_i$  is less than  $3^k$  for  $s < i \leq r$ . Let  $T = \{i \in \{1, 2, \dots, n\} \mid i \text{ occurs in the permutation } \tau_1 \tau_2 \dots \tau_s\}$ . Let  $M_1 = M \cap \sum_{i \in T} X_i$  and  $N_1 = M \cap \sum_{i \notin T} X_i$ . We claim that  $M = M_1 \oplus N_1$

and that  $M_1$  is invariant under  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ , where each  $\alpha_i$  belongs to  $\{\omega, \omega^2\}$ . Let  $(x, y) \in M$ , where  $x \in \sum_{i \in T} X_i$ ,  $y \in \sum_{i \notin T} X_i$ . Since

$$\sigma^{3^k} = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n),$$

where  $\alpha_i$  belongs to  $\{\omega, \omega^2\}$  for  $i \in T$  and  $\alpha_i = 1$  for  $i \notin T$ , it follows that,  $(x, y) + \sigma^{3^k}(x, y) + (\sigma^{3^k})^2(x, y) = (0, y)$  belongs to  $M$ . Hence  $(x, 0)$  belongs to  $M$  as well. Thus  $M = M_1 \oplus N_1$ . Clearly  $M_1$  is invariant under  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ ,  $\alpha_i$  being in  $\{\omega, \omega^2\}$ . Since  $\sigma/N_1$  is perfect, by

repeating the above argument we obtain a similar decomposition of  $N_1: N_1 = M_2 \oplus N_2$ . This process terminates in a finite number of steps and we obtain a decomposition  $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$ , where each  $M_j$  is invariant under  $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ ,  $\alpha_i$  being in  $\{\omega, \omega^2\}$ .

## §5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

**5.1. PROPOSITION.** *Let  $L$  be a unimodular  $\mathbf{Z}$ -lattice of type  $nD_4$  such that  $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$ . If  $L$  admits a perfect isometry, then there exists an isometry  $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$  on  $\mathcal{H}^{*n}$ , where  $\delta_i$  is the isometry on  $\mathcal{H}^*$  given by left multiplication by  $\xi$  or right multiplication by  $\bar{\xi}$  such that  $L$  is invariant under  $\delta$ .*

*Proof.* Let  $\sigma$  be a perfect isometry of  $(L, \text{Tr} \circ h)$ . Then  $\sigma$  induces an automorphism of  $\mathcal{H}^n$  and extends naturally to a perfect isometry of  $\mathcal{H}^{*n}$ . In view of ([K], p. 179),  $\eta(\sigma)$  is a perfect isomorphism of  $\mathbf{F}_4^n$ , leaving  $\eta(L)$  invariant. Therefore by Proposition 4.7 there exists  $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$  with  $\alpha_i$  in  $\{\omega, \omega^2\}$  such that  $\eta(L)$  is invariant under  $\alpha$ . Let  $\delta_i$  denote left multiplication on  $\mathcal{H}^*$  by  $\xi = (1 + i + j + k)/2$  if  $\alpha_i = \omega$  and right multiplication by  $\bar{\xi} = (1 - i - j - k)/2$ , if  $\alpha_i = \omega^2$ . Let  $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ . Since  $\delta$  induces an isometry of  $\mathcal{H}^{*n}$  which fixes  $\mathcal{H}^n$  and  $\eta(\delta) = \alpha$  leaves  $\eta(L)$  invariant it follows that  $\delta$  leaves  $L$  invariant.

**5.2. THEOREM.** *Let  $(L, S)$  be an unimodular  $\mathbf{Z}$ -lattice of type  $nD_4$ . Then,  $L$  has a perfect isometry if and only if there exists an  $\mathcal{H}$ -lattice  $(L', S')$  such that  $L \simeq L'$ .*

*Proof.* Clearly every  $\mathcal{H}$ -lattice admits a perfect isometry (3.2). Conversely let  $(L, S)$  be a  $\mathbf{Z}$ -lattice of type  $nD_4$ , which admits a perfect isometry. In view of Proposition 2.1, we can assume that  $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$  and  $S = \text{Tr} \circ h$ . By Proposition 4.7 there exists a subset  $T$  of  $\{1, 2, \dots, n\}$  such that  $L$  is invariant under  $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$ , where  $\delta_i$  is left multiplication by  $\xi$  for  $i \in T$  and  $\delta_i$  is right multiplication by  $\bar{\xi}$  for  $i \notin T$ . Let  $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$  be defined by  $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$  where  $f_i = \text{id}$  for  $i \in T$  and  $f_i$  is the involution on  $\mathcal{H}$  for  $i \notin T$ . Then it is easy to check that  $f$  is an isometry of  $(L, \text{Tr} \circ h)$  onto  $(L', S')$  where,  $L' = f(L)$ , and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$