

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	39 (1993)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS
Autor:	Shastri, Parvati
Kapitel:	§4. Automorphisms of the root System D_4 and perfect isometries
DOI:	https://doi.org/10.5169/seals-60415

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. Note that every perfect isometry σ of \mathcal{H} extends naturally to a perfect isometry of \mathcal{H}^* , inducing a perfect \mathbf{F}_2 -isomorphism $\eta(\sigma)$ of $\mathcal{H}^*/\mathcal{H}$, η denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it corresponds to multiplication by ω , where ω denotes a primitive element of \mathbf{F}_4 over \mathbf{F}_2 .*

Proof. An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of \mathbf{F}_4 , corresponds to multiplication by ω , ω being as above.

3.6. PROPOSITION. *Let L be a \mathbf{Z} -lattice such that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$. If L is an \mathcal{H} -lattice, then L has a perfect isometry, which corresponds to multiplication by ω , on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$.*

Proof. Multiplication by ξ is a perfect isometry of \mathcal{H}^n which extends naturally to a perfect isometry of \mathcal{H}^{*n} . Clearly the induced map on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$ is multiplication by ω . Since L is an \mathcal{H} -module, it preserves L as well.

In particular,

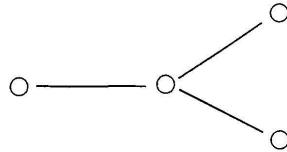
3.7. COROLLARY. *Every \mathcal{H} -lattice $(L, Tr \circ h)$ of type nD_4 has a perfect isometry.*

It is but natural to ask whether every \mathbf{Z} -lattice of type nD_4 which has a perfect isometry necessarily admits the structure of an \mathcal{H} -lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system nD_4 .

§4. AUTOMORPHISMS OF THE ROOT SYSTEM nD_4 AND PERFECT ISOMETRIES

For any root system R , let $\mathcal{W}(R)$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathcal{W}(R)$ is a normal subgroup of $Aut R$, which preserves every \mathbf{Z} -lattice L such that $\mathbf{Z}R \subseteq L \subseteq \mathbf{Z}R^*$. We thus get a natural map $\eta: Aut R/\mathcal{W}(R) \rightarrow Aut_{\mathbf{Z}}(\mathbf{Z}R^*/\mathbf{Z}R)$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.

An element σ in $Aut(R)/\mathcal{W}(R)$ preserves L if and only if $\eta(\sigma)$ preserves the corresponding subgroup $\eta(L)$ of $\mathbf{Z}R^\#/\mathbf{Z}R$. If $R = D_4$, $Aut R = \mathcal{W}(R) \underset{s}{\times} S_3$, where, $\underset{s}{\times}$ denotes the semi direct product and S_3 is the automorphism group of the associated Dynkin diagram:



Consequently, for $R = nD_4$, $Aut R/\mathcal{W}(R) \simeq S_3^n \underset{s}{\times} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\times} S_n$. Thus the elements of $Aut R/\mathcal{W}(R)$ are “monomial matrices” where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. It acts naturally on $(\mathbf{Z}D_4^\#)^n/\mathbf{Z}D_4^n$. In view of the identification of $\mathbf{Z}D_4^\#/\mathbf{Z}D_4 \simeq \mathcal{H}^*/\mathcal{H}$, we have the following proposition.

4.1. PROPOSITION.

- (a) $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n) \simeq S_3^n \underset{s}{\times} S_n \simeq (GL_2(\mathbf{F}_2))^n \underset{s}{\times} S_n$.
- (b) If U denotes the group of units of \mathcal{H} , then U is a subgroup of $Aut \mathcal{H}$ and $U/(\mathcal{W}(\mathcal{H}) \cap U) \simeq \{1, \omega, \omega^2\}$, where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.
- (c) The conjugation in \mathcal{H} belongs to the Weyl group $\mathcal{W}(\mathcal{H})$.

Proof. (a) This statement is an immediate consequence of the identification $\mathbf{Z}D_4 \simeq \mathcal{H}$.

- (b) By (a), $Aut \mathcal{H}/\mathcal{W}(\mathcal{H}) \simeq S_3 \simeq GL_2(\mathbf{F}_2)$. Since $\eta(U) = \{1, \omega, \omega^2\}$, (b) follows.
- (c) The conjugation in \mathcal{H} is a product of reflections defined by i, j and k .

We now consider the perfect isomorphisms of $(\mathcal{H}^*)^n/\mathcal{H}^n$ arising out of $Aut(\mathcal{H}^n)/\mathcal{W}(\mathcal{H}^n)$. We begin by fixing the following notation:

Let $V = \mathbf{F}_4^n = X_1 \perp X_2 \perp \dots \perp X_n$ with respect to the standard hermitian form on V , where $X_i \simeq \mathbf{F}_4 = \mathbf{F}_2 \oplus \mathbf{F}_2 = \{0, 1, \omega, \omega^2\}$. Let G denote the group of all $n \times n$ monomial matrices with entries in $M_2(\mathbf{F}_2)$, where each row and each column consists of exactly one element of $GL_2(\mathbf{F}_2)$. Note that every element of G can be uniquely expressed as $\alpha \cdot \tau$, where α is the diagonal matrix $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, with α_i in $GL_2(\mathbf{F}_2)$ and τ is an $n \times n$ permutation matrix. We have,

4.2. LEMMA. *Let σ belonging to G be perfect and let $X = X_i$ for some i . Let m be the smallest positive integer for which σ^m maps X onto itself. Then σ^m/X is perfect.*

Proof. The idea of the proof is similar to ([K], Prop. 2). We show that $(1 - \sigma^m)/X$ is surjective. Let $M = \sum_{0 \leq i \leq m-1} \sigma^i(X)$. Then σ leaves M invariant. Therefore σ is a perfect isomorphism of M . Hence $(1 - \sigma)/M: M \rightarrow M$ is surjective. Let x be an element of X . Since, $(x, 0, \dots, 0)$ belongs to M , there exists an element y in M such that $(1 - \sigma)(y) = (x, 0, \dots, 0)$. Let $y = (y_0, y_1, \dots, y_{m-1})$, where y_i belongs to $\sigma^i(X)$. Then,

$$(1 - \sigma)(y) = (y_0 - \sigma(y_{m-1}), y_1 - \sigma(y_0), \dots, y_{m-1} - \sigma(y_{m-2})).$$

Hence, $y_0 - \sigma(y_{m-1}) = x, y_1 = \sigma(y_0), \dots, y_{m-1} = \sigma(y_{m-2})$. Further, $\sigma(y_{m-1}) = \sigma^2(y_{m-2}) = \dots = \sigma^m(y_0)$. Thus $(1 - \sigma^m)(y_0) = x$. This implies that $(1 - \sigma^m)/X$ is surjective.

4.3. COROLLARY. *Let σ be an element of G which is perfect. Suppose that $\sigma = \alpha \cdot \tau$, where $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, $\alpha_i \in GL_2(\mathbf{F}_2)$, $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, and τ_i are disjoint cyclic permutations of length n_i . Let T_i denote the set of indices belonging to the permutation τ_i . Then $(\sigma)^{n_i}/X_j$ is perfect for every j belonging to T_i .*

Proof. Note that for every j belonging to T_i , n_i is the smallest positive integer such that $(\sigma)^{n_i}$ maps X_j onto itself.

4.4. COROLLARY. *If σ is as above, then $(\sigma)^{n_i}/X_j$ corresponds to multiplication by ω or ω^2 , for every j belonging to T_i .*

Proof. Follows from Corollary 4.3, and Lemma 3.5.

4.5. COROLLARY. *If σ is as above, and $X^{(i)} = \sum_{j \in T_i} X_j$, then $(\sigma)^{n_i}/X^{(i)}$ is the matrix $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$, where α_j belongs to $\{\omega, \omega^2\}$.*

Proof. Clear from Corollary 4.4.

4.6. PROPOSITION. *Let σ be an element of G which is perfect and let $\sigma = \alpha \cdot \tau$, where α and τ are as in Corollary 4.4. Then there exists an integer $l \geq 1$, such that σ^l is perfect and $\sigma^l = \beta \cdot \tau'$, where β is the matrix $\text{diag}(\beta_1, \dots, \beta_j, \dots, \beta_n)$, with β_j in $GL_2(\mathbf{F}_2)$ and τ' is a product of disjoint cyclic permutations τ_i of length 3^{k_i} .*

Proof. Let $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, where τ_i are disjoint cyclic permutations of length $n_i = 3^{k_i} \cdot l_i$, with $(3, l_i) = 1$. Let l denote the least common multiple of the l_i . We show that σ^l is perfect. By Corollary 4.5, σ^{n_i}/X_j is multiplication by ω or ω^2 for every j belonging to T_i . This implies that $(\sigma)^{n_i l / l_i}/X_j$ corresponds to multiplication by ω or ω^2 for every such j , since $(l/l_i, 3) = 1$ and ω is an element of order 3. Hence, $(\sigma^l)^{3^{k_i}}/X^{(i)}$ is the matrix $\text{diag}(\alpha_1, \dots, \alpha_j, \dots, \alpha_{n_i})$ where α_j belongs to $\{\omega, \omega^2\}$. Clearly this implies that $\sigma^l/X^{(i)}$ has no nontrivial fixed point. Since T_i are disjoint, it follows that σ^l has no nontrivial fixed point and hence σ^l is perfect. Obviously σ^l has the required property and the proposition follows.

Now, let M be an \mathbf{F}_2 -linear subspace of V , which is invariant under a perfect isomorphism σ belonging to G . By the previous proposition, we can assume, by replacing σ by σ^m , that M is invariant under $\sigma = \alpha \cdot \tau$, where α is as in Corollary 4.4 and $\tau = \tau_1 \cdot \tau_2 \dots \tau_r$, τ_i being cyclic permutations of length 3^{k_i} .

4.7. PROPOSITION. *If M is an \mathbf{F}_2 -linear subspace of V which has a perfect isomorphism σ belonging to G , then M is invariant under the action of a diagonal matrix, $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ where each α_i belongs to $\{\omega, \omega^2\}$.*

Proof. By replacing σ by a suitable power we may assume that

$$\sigma = \text{diag}(\beta_1, \dots, \beta_i, \dots, \beta_n) \tau_1 \tau_2 \dots \tau_r$$

where β_i belongs to $GL_2(\mathbf{F}_2)$ for every i and τ_i are disjoint cyclic permutations of length 3^{k_i} . Further, since disjoint cycles commute we may assume that the length of τ_i is 3^k for $1 \leq i \leq s$ and the length of τ_i is less than 3^k for $s < i \leq r$. Let $T = \{i \in \{1, 2, \dots, n\} \mid i \text{ occurs in the permutation } \tau_1 \tau_2 \dots \tau_s\}$. Let $M_1 = M \cap \sum_{i \in T} X_i$ and $N_1 = M \cap \sum_{i \notin T} X_i$. We claim that $M = M_1 \oplus N_1$

and that M_1 is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, where each α_i belongs to $\{\omega, \omega^2\}$. Let $(x, y) \in M$, where $x \in \sum_{i \in T} X_i$, $y \in \sum_{i \notin T} X_i$. Since

$$\sigma^{3^k} = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n),$$

where α_i belongs to $\{\omega, \omega^2\}$ for $i \in T$ and $\alpha_i = 1$ for $i \notin T$, it follows that, $(x, y) + \sigma^{3^k}(x, y) + (\sigma^{3^k})^2(x, y) = (0, y)$ belongs to M . Hence $(x, 0)$ belongs to M as well. Thus $M = M_1 \oplus N_1$. Clearly M_1 is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$. Since σ/N_1 is perfect, by

repeating the above argument we obtain a similar decomposition of $N_1: N_1 = M_2 \oplus N_2$. This process terminates in a finite number of steps and we obtain a decomposition $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_j is invariant under $\text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$, α_i being in $\{\omega, \omega^2\}$.

§5. MAIN THEOREM AND EXAMPLES

In this final section we prove our main results 5.2, 5.3 and give some examples. We begin with,

5.1. PROPOSITION. *Let L be a unimodular \mathbf{Z} -lattice of type nD_4 such that $\mathcal{H}^n \subset L \subset \mathcal{H}^{*n}$. If L admits a perfect isometry, then there exists an isometry $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$ on \mathcal{H}^{*n} , where δ_i is the isometry on \mathcal{H}^* given by left multiplication by ξ or right multiplication by $\bar{\xi}$ such that L is invariant under δ .*

Proof. Let σ be a perfect isometry of $(L, Tr \circ h)$. Then σ induces an automorphism of \mathcal{H}^n and extends naturally to a perfect isometry of \mathcal{H}^{*n} . In view of ([K], p. 179), $\eta(\sigma)$ is a perfect isomorphism of F_4^n , leaving $\eta(L)$ invariant. Therefore by Proposition 4.7 there exists $\alpha = \text{diag}(\alpha_1, \dots, \alpha_i, \dots, \alpha_n)$ with α_i in $\{\omega, \omega^2\}$ such that $\eta(L)$ is invariant under α . Let δ_i denote left multiplication on \mathcal{H}^* by $\xi = (1 + i + j + k)/2$ if $\alpha_i = \omega$ and right multiplication by $\bar{\xi} = (1 - i - j - k)/2$, if $\alpha_i = \omega^2$. Let $\delta = \text{diag}(\delta_1, \dots, \delta_i, \dots, \delta_n)$. Since δ induces an isometry of \mathcal{H}^{*n} which fixes \mathcal{H}^n and $\eta(\delta) = \alpha$ leaves $\eta(L)$ invariant it follows that δ leaves L invariant.

5.2. THEOREM. *Let (L, S) be an unimodular \mathbf{Z} -lattice of type nD_4 . Then, L has a perfect isometry if and only if there exists an \mathcal{H} -lattice (L', S') such that $L \simeq L'$.*

Proof. Clearly every \mathcal{H} -lattice admits a perfect isometry (3.2). Conversely let (L, S) be a \mathbf{Z} -lattice of type nD_4 , which admits a perfect isometry. In view of Proposition 2.1, we can assume that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ and $S = Tr \circ h$. By Proposition 4.7 there exists a subset T of $\{1, 2, \dots, n\}$ such that L is invariant under $\delta = (\delta_1, \dots, \delta_i, \dots, \delta_n)$, where δ_i is left multiplication by ξ for $i \in T$ and δ_i is right multiplication by $\bar{\xi}$ for $i \notin T$. Let $f: \mathcal{H}^n \rightarrow \mathcal{H}^n$ be defined by $f = \text{diag}(f_1, \dots, f_i, \dots, f_n)$ where $f_i = \text{id}$ for $i \in T$ and $f_i = \text{the involution on } \mathcal{H}$ for $i \notin T$. Then it is easy to check that f is an isometry of $(L, Tr \circ h)$ onto (L', S') where, $L' = f(L)$, and,

$$S'(x, y) = \sum_{i \in T} (x_i \bar{y}_i + y_i \bar{x}_i) + \sum_{i \notin T} (\bar{x}_i y_i + \bar{y}_i x_i).$$