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2.3. PROPOSITION.

(a) *Let* $\mathscr{H}^n \subseteq L \subseteq \mathscr{H}^{*n}$ *be a* **Z**-*module. Then* $(L, Tr \circ h)$ *is integral if and only if* $\eta(L)$ *is a totally isotropic subspace of the symmetric bilinear space* $(\mathbf{F}_4^n, Tr \circ \eta(h))$, *where* $\eta(h)$ *is the standard hermitian form on* $\mathbf{F}_4^n$.

(b) *The* **Z**-*lattice* $(L, Tr \circ h)$ *is unimodular if and only if* $\eta(L)$ *is a maximal totally isotropic subspace of* $(\mathbf{F}_4^n, Tr \circ \eta(h))$.

Proof. (a) This follows easily from 2.2.

(b) This follows from (a), since L is unimodular if and only if L is maximal integral.

§3. PERFECT ISOMETRIES OF $\mathscr{H}$-LATTICES

In this section we show that certain special class of **Z**-lattices admit perfect isometries. We begin with the following definition.

3.1. *Definition.* A **Z**-lattice (L, b) is called an *$\mathscr{H}$-lattice* if L is an $\mathscr{H}$-module and $b = Tr \circ h$ for some hermitian form h.

3.2. PROPOSITION. *Every $\mathscr{H}$-lattice has a perfect isometry.*

Proof. Let $(L, Tr \circ h)$ be an $\mathscr{H}$-lattice. Let $\sigma : L \to L$ denote left (or right) multiplication by ξ where ξ is one of the units $(1 \pm i \pm j \pm k)/2$. Then,

$$Tr \circ h(\sigma(x), \sigma(y)) = Tr \circ h(\xi x, \xi y) = Tr(\xi h(x, y)\bar{\xi})$$
$$= \xi h(x, y)\bar{\xi} + \xi\overline{h(x, y)}\bar{\xi} = \xi(h(x, y) + \overline{h(x, y)})\bar{\xi} = \xi\bar{\xi}(h(x, y) + \overline{h(x, y)})$$
$$= h(x, y) + \overline{h(x, y)} = Tr \circ h(x, y)\ .$$

Therefore σ is an isometry. Since the minimal polynomial of σ is $x^2 - x + 1$, $\det(1 - \sigma) = 1$ and hence σ is perfect.

As a special case of this we have:

3.3. COROLLARY. *The $\mathscr{H}$-lattice* $(\mathscr{H}, Tr \circ h)$ *has a perfect isometry.*

3.4. PROPOSITION. *Every perfect isometry of* $(\mathscr{H}, Tr \circ h)$ *induces a perfect* $\mathbf{F}_2$-*isomorphism of* $\mathscr{H}^*/\mathscr{H} = \mathbf{F}_4$, *which corresponds to multiplication by* ω, *where* $\mathbf{F}_2(\omega) = \mathbf{F}_4$.

Proof. Note that every perfect isometry σ of $\mathcal{H}$ extends naturally to a perfect isometry of $\mathcal{H}^*$, inducing a perfect $\mathbf{F}_2$-isomorphism $\eta(\sigma)$ of $\mathcal{H}^*/\mathcal{H}$, η denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it corresponds to multiplication by ω, where ω denotes a primitive element of $\mathbf{F}_4$ over $\mathbf{F}_2$.*

Proof. An $\mathbf{F}_2$-linear isomorphism of $\mathbf{F}_4$ is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of $\mathbf{F}_4$, corresponds to multiplication by ω, ω being as above.

3.6. PROPOSITION. *Let L be a $\mathbf{Z}$-lattice such that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$. If L is an $\mathcal{H}$-lattice, then L has a perfect isometry, which corresponds to multiplication by ω, on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$.*

Proof. Multiplication by ξ is a perfect isometry of $\mathcal{H}^n$ which extends naturally to a perfect isometry of $\mathcal{H}^{*n}$. Clearly the induced map on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$ is multiplication by ω. Since L is an $\mathcal{H}$-module, it preserves L as well.

In particular,

3.7. COROLLARY. *Every $\mathcal{H}$-lattice $(L, Tr \circ h)$ of type $n\mathrm{D}_4$ has a perfect isometry.*

It is but natural to ask whether every $\mathbf{Z}$-lattice of type $n\mathrm{D}_4$ which has a perfect isometry necessarily admits the structure of an $\mathcal{H}$-lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system $n\mathrm{D}_4$.

§4. AUTOMORPHISMS OF THE ROOT SYSTEM $n\mathrm{D}_4$ AND PERFECT ISOMETRIES

For any root system R, let $\mathcal{W}(\mathrm{R})$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathcal{W}(\mathrm{R})$ is a normal subgroup of $Aut\,\mathrm{R}$, which preserves every $\mathbf{Z}$-lattice L such that $\mathbf{Z}\mathrm{R} \subseteq L \subseteq \mathbf{Z}\mathrm{R}^{\#}$. We thus get a natural map $\eta: Aut\,\mathrm{R}/\mathcal{W}(\mathrm{R}) \to Aut_{\mathbf{Z}}(\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R})$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.