

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS
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Kapitel: §3. Perfect isometries of H -lattices
DOI: <https://doi.org/10.5169/seals-60415>

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2.3. PROPOSITION.

(a) Let $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$ be a \mathbf{Z} -module. Then $(L, \text{Tr} \circ h)$ is integral if and only if $\eta(L)$ is a totally isotropic subspace of the symmetric bilinear space $(\mathbf{F}_4^n, \text{Tr} \circ \eta(h))$, where $\eta(h)$ is the standard hermitian form on \mathbf{F}_4^n .

(b) The \mathbf{Z} -lattice $(L, \text{Tr} \circ h)$ is unimodular if and only if $\eta(L)$ is a maximal totally isotropic subspace of $(\mathbf{F}_4^n, \text{Tr} \circ \eta(h))$.

Proof. (a) This follows easily from 2.2.

(b) This follows from (a), since L is unimodular if and only if L is maximal integral.

§3. PERFECT ISOMETRIES OF \mathcal{H} -LATTICES

In this section we show that certain special class of \mathbf{Z} -lattices admit perfect isometries. We begin with the following definition.

3.1. *Definition.* A \mathbf{Z} -lattice (L, b) is called an \mathcal{H} -lattice if L is an \mathcal{H} -module and $b = \text{Tr} \circ h$ for some hermitian form h .

3.2. PROPOSITION. Every \mathcal{H} -lattice has a perfect isometry.

Proof. Let $(L, \text{Tr} \circ h)$ be an \mathcal{H} -lattice. Let $\sigma: L \rightarrow L$ denote left (or right) multiplication by ξ where ξ is one of the units $(1 \pm i \pm j \pm k)/2$. Then,

$$\begin{aligned} \text{Tr} \circ h(\sigma(x), \sigma(y)) &= \text{Tr} \circ h(\xi x, \xi y) = \text{Tr}(\xi h(x, y) \bar{\xi}) \\ &= \xi h(x, y) \bar{\xi} + \xi \overline{h(x, y)} \bar{\xi} = \xi (h(x, y) + \overline{h(x, y)}) \bar{\xi} = \xi \bar{\xi} (h(x, y) + \overline{h(x, y)}) \\ &= h(x, y) + \overline{h(x, y)} = \text{Tr} \circ h(x, y). \end{aligned}$$

Therefore σ is an isometry. Since the minimal polynomial of σ is $x^2 - x + 1$, $\det(1 - \sigma) = 1$ and hence σ is perfect.

As a special case of this we have:

3.3. COROLLARY. The \mathcal{H} -lattice $(\mathcal{H}, \text{Tr} \circ h)$ has a perfect isometry.

3.4. PROPOSITION. Every perfect isometry of $(\mathcal{H}, \text{Tr} \circ h)$ induces a perfect \mathbf{F}_2 -isomorphism of $\mathcal{H}^*/\mathcal{H} = \mathbf{F}_4$, which corresponds to multiplication by ω , where $\mathbf{F}_2(\omega) = \mathbf{F}_4$.

Proof. Note that every perfect isometry σ of \mathcal{H} extends naturally to a perfect isometry of \mathcal{H}^* , inducing a perfect \mathbf{F}_2 -isomorphism $\eta(\sigma)$ of $\mathcal{H}^*/\mathcal{H}$, η denoting the induced map on the quotient. The proof of the proposition is complete in view of the following simple lemma.

3.5. LEMMA. *An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it corresponds to multiplication by ω , where ω denotes a primitive element of \mathbf{F}_4 over \mathbf{F}_2 .*

Proof. An \mathbf{F}_2 -linear isomorphism of \mathbf{F}_4 is perfect if and only if it has no fixed point other than the trivial element. Since, $GL_2(\mathbf{F}_2) \simeq S_3$, it is easy to see that every perfect isomorphism of \mathbf{F}_4 , corresponds to multiplication by ω , ω being as above.

3.6. PROPOSITION. *Let L be a \mathbf{Z} -lattice such that $\mathcal{H}^n \subseteq L \subseteq \mathcal{H}^{*n}$. If L is an \mathcal{H} -lattice, then L has a perfect isometry, which corresponds to multiplication by ω , on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$.*

Proof. Multiplication by ξ is a perfect isometry of \mathcal{H}^n which extends naturally to a perfect isometry of \mathcal{H}^{*n} . Clearly the induced map on the quotient $\mathcal{H}^{*n}/\mathcal{H}^n$ is multiplication by ω . Since L is an \mathcal{H} -module, it preserves L as well.

In particular,

3.7. COROLLARY. *Every \mathcal{H} -lattice $(L, Tr \circ h)$ of type nD_4 has a perfect isometry.*

It is but natural to ask whether every \mathbf{Z} -lattice of type nD_4 which has a perfect isometry necessarily admits the structure of an \mathcal{H} -lattice. We shall show that this is indeed true. For doing this we need to recall some basic facts on the automorphisms of the root system nD_4 .

§4. AUTOMORPHISMS OF THE ROOT SYSTEM nD_4 AND PERFECT ISOMETRIES

For any root system R , let $\mathcal{W}(R)$ denote the Weyl group of R (i.e. the group generated by the reflections defined by the roots). Then $\mathcal{W}(R)$ is a normal subgroup of $Aut R$, which preserves every \mathbf{Z} -lattice L such that $\mathbf{Z}R \subseteq L \subseteq \mathbf{Z}R^\#$. We thus get a natural map $\eta: Aut R/\mathcal{W}(R) \rightarrow Aut_{\mathbf{Z}}(\mathbf{Z}R^\#/\mathbf{Z}R)$. In view of ([H], p. 72; [C-S], p. 432) this is an injection.