6. Applications

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **39 (1993)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

(1) Any equation of the form

$$x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x + a_{n+1} = 0$$

with coefficients in R has a root in R.

(2) The value group G satisfies $G = pG$.

Also if G is discrete of arbitrary rank and char F = char R, then the extension in unique [6]. But Kaplansky gives examples where the extension is not unique. The exact conditions under which the extension is unique are not known.

6. Applications

One application of Theorem 2 is to the problem of "glueing" two valued fields. (This result can also be proved directly without the use of Mal'cev-Neumann fields; it is equivalent to Exercise 2 for §2 in Chapter VI of [2]. Our method has the advantage of showing that the value group of the composite field can be contained in any divisible value group large enough to contain the value groups of the fields to be glued.)

PROPOSITION 8. *Suppose* E, F, F' *are valued fields and that we are given embeddings of valued fields* $\phi : E \to F, \phi' : E \to F'$. *Then there exist a Mal'cev-Neumann field* L *(or* K*) and embeddings of valued fields* $\Phi : F \to L, \Phi' : F' \to L$ *such that* $\Phi \circ \phi = \Phi' \circ \phi'$.

Proof. By the glueing theorem for ordered groups [14], we can assume the value groups of F and F' are contained in a single ordered group G. Also we can assume that their residue fields are contained in a field R. Moreover, we may assume G is divisible and R is algebraically closed. Then E can be embedded as a valued subfield of a power series field L (or K) with value group G and residue field R, by Corollary 5. Finally, Theorem 2 gives us the desired embeddings Φ, Φ'. □

Remark. Transfinite induction can be used to prove the analogous result for glueing an arbitrary collection of valued fields.

Since a non-archimedean absolute value on a field can be interpreted as a valuation with value group contained in $\mathbf{R}$, we can specialize the results of Section 5 to get results about fields with non-archimedean absolute values. For example, Corollary 5 implies the following, which may be considered the non-archimedean analogue of Ostrowski's theorem that any field with an archimedean absolute value can be embedded in $\mathbf{C}$ with its usual absolute value (or one equivalent).

PROPOSITION 9. *Let* $(F, |\,|)$ *be a field with a non-archimedean absolute value, and suppose the residue field is contained in the algebraically closed field* R. *Define* K *and* L *as the Mal'cev-Neumann fields with value group* $\mathbf{R}$ *and residue field* R. *(Define the p-adic Mal'cev-Neumann field* L *only if* $\operatorname{char} R > 0$.*) The valuations on* K *and* L *induce corresponding absolute values. Then there exists an absolute value-preserving embedding of fields* $\phi : F \to K$ *or* $\phi : F \to L$, *depending on if the restriction of* $|\,|$ *to the minimal subfield of* F *is the trivial absolute value (on* $\mathbf{Q}$ *or* $\mathbf{F}_p$*) or the p-adic absolute value on* $\mathbf{Q}$.

Similarly, Proposition 8 above gives a glueing proposition for non-archimedean absolute values. In fact, this result holds for archimedean absolute values as well, in light of Ostrowski's theorem.

7. EXAMPLE: THE MAXIMALLY COMPLETE IMMEDIATE EXTENSION OF $\bar{\mathbf{Q}}_p$

For this section, (L, v) will denote the p-adic Mal'cev-Neumann field having value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$. We have a natural embedding of $\mathbf{Q}_p$ into L. By Corollary 4, L is algebraically closed, so this embedding extends to an embedding of $\bar{\mathbf{Q}}_p$ into L (which is unique up to automorphisms of $\bar{\mathbf{Q}}_p$ over $\mathbf{Q}_p$.) In fact this embedding is continuous, since there is a unique valuation on $\bar{\mathbf{Q}}_p$ extending the p-adic valuation on $\mathbf{Q}_p$. Since $\bar{\mathbf{Q}}_p$ has value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$, L is an immediate extension of $\bar{\mathbf{Q}}_p$. By Corollary 6, L is in fact the unique maximally complete immediate extension of $\bar{\mathbf{Q}}_p$. Also, any valued field (F, w) of characteristic 0 satisfying

(1) The restriction of w to $\mathbf{Q}$ is the p-adic valuation;

(2) The value group is contained in $\mathbf{Q}$;

(3) The residue field is contained in $\bar{\mathbf{F}}_p$;

can be embedded in L, by Corollary 5. For example, the completion $\mathbf{C}_p$ of $\bar{\mathbf{Q}}_p$ can be embedded in L. (This could also be proved by noting that L is complete by Corollary 4.)

We will always use as the set S of representatives for $\bar{\mathbf{F}}_p$ the primitive k^{th} roots of 1, for all k not divisible by p, and 0. Then the elements of L have the form $\sum_g \alpha_g p^g$ for some primitive k^{th} roots α_g of 1, where the exponents form a well-ordered subset of $\mathbf{Q}$. In particular, the elements of $\bar{\mathbf{Q}}_p$ can be expressed in this form. This was first discovered by Lampert [9].