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4. p -ADIC MAL'CEV-NEUMANN FIELDS

To construct analogous examples of characteristic zero whose residue field has nonzero characteristic requires a more complicated construction. First we recall two results about complete discrete valuation rings. For proofs, see [17], pp. 32-34.

A valued field (F, v) is called *discrete* if $v(F) = \mathbf{Z}$.

PROPOSITION 1. *If R is a perfect field of characteristic $p > 0$, then there exists a unique field R' of characteristic 0 with a discrete valuation v such that the residue field is R , $v(p) = 1 \in \mathbf{Z}$, and R' is complete with respect to v . (The valuation ring A of R' is called the ring of Witt vectors with coefficients in R .)*

For example, if $R = \mathbf{F}_p$, then $R' = \mathbf{Q}_p$ with the p -adic valuation.

PROPOSITION 2. *Suppose F is field with a discrete valuation v , and $t \in F$ satisfies $v(t) = 1$. Let $S \subset F$ be a set of representatives for the residue classes with $0 \in S$. Then every element $x \in F$ can be written uniquely as $\sum_{m \in \mathbf{Z}} x_m t^m$, where $x_m \in S$ for each m , and $x_m = 0$ for all sufficiently negative m . Conversely, if F is complete, every such series defines an element of F .*

Now for the construction. Let R be a perfect field of characteristic p , and let G be an ordered group containing \mathbf{Z} as a subgroup, or equivalently with a distinguished positive element. (When we eventually define our valuation v , this element 1 $\in G$ will be $v(p)$.) Let A be the valuation ring of the valued field (R', v') given by Proposition 1.

What we want is to have the indeterminate t stand for p in elements of $A((G))$, so we get elements of the form $\sum_{g \in G} \alpha_g p^g$. The problem is that some elements of $A((G))$, like $-p + t^1$, "should be" zero. So what we do is to take a quotient $A((G))/N$ where $N \subset A((G))$ is the ideal of elements that "should be" zero.

We say that $\alpha = \sum_g \alpha_g t^g \in A((G))$ is a *null series* if for all $g \in G$, $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = 0$ in R' . (Recall that we fixed a copy of \mathbf{Z} in G .) Note that $\alpha_{g+n} = 0$ for sufficiently negative n , since otherwise $\text{Supp } \alpha$ would not be well-ordered. Also, $v'(\alpha_{g+n} p^n) \geq n$, so $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n$ always converges in R' . Let N be the set of null series.

PROPOSITION 3. N is an ideal of $A((G))$.

Proof. Clearly N is an additive subgroup. Let $G' \subset G$ be a set of coset representatives for G/\mathbf{Z} . Suppose $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, $\beta = \sum_{h \in G} \beta_h t^h \in N$, and $\alpha\beta = \sum_{j \in G} \gamma_j t^j$. Then for each $j \in G$,

$$\begin{aligned} \sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n &= \sum_{\substack{g+h=j+n \\ n \in \mathbf{Z}}} \alpha_g \beta_h p^n \\ &= \sum_{\substack{h' \in G' \\ l, m \in \mathbf{Z}}} (\alpha_{j-h'+l} p^l) (\beta_{h'+m} p^m) \end{aligned}$$

(We write $h = h' + m$ with $h' \in G'$ and let $l = n - m$.)

Since $\beta \in N$, $\sum_{m \in \mathbf{Z}} \beta_{h'+m} p^m = 0$ for each $h' \in G'$, so we get $\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = 0$. (These infinite series manipulations in R' are valid, because for each $i \in \mathbf{Z}$, only finitely many terms have valuation less than i , since each γ_{j+n} is a finite sum of products $\alpha_g \beta_h$.) Hence N is an ideal. \square

Define the p -adic Mal'cev-Neumann field L as $A((G))/N$.

PROPOSITION 4. Let $S \subset A$ be a set of representatives for the residue classes of A , with $0 \in S$. Then any element $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$ is equivalent modulo N to a element $\beta = \sum_{g \in G} \beta_g t^g$ with each β_g in S . Moreover, β is unique.

Proof. Let $G' \subset G$ be a set of coset representatives for G/\mathbf{Z} . For each $g \in G'$, we may write

$$\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = \sum_{n \in \mathbf{Z}} \beta_{g+n} p^n$$

with $\beta_{g+n} \in S$, by Proposition 2. (This is possible since R' is complete with respect to its discrete valuation.) Then $\beta = \sum_{g \in G'} \sum_{n \in \mathbf{Z}} \beta_{g+n} t^n$ is a well-defined element of $A((G))$, since $\text{Supp}(\beta) \subseteq (\text{Supp} \alpha) + \mathbf{N}$, which is well-ordered by part 2 of Lemma 1. Finally $\alpha - \beta \in N$, by definition of N . The uniqueness follows from the uniqueness in Proposition 2. \square

COROLLARY 3. $L = A((G))/N$ is a field.

Proof. The previous proposition shows that any $\alpha \in A((G))$ is equivalent modulo N to 0 or an element which is a unit in $A((G))$ by Corollary 1. \square

Proposition 4 allows us to write an element of L uniquely (and somewhat carelessly) as $\beta = \sum_{g \in G} \beta_g p^g$, with $\beta_g \in S$. Thus given S , we can speak of $\text{Supp}(\beta)$ for $\beta \in L$. Define $v: L \rightarrow G_\infty$ by $v(\beta) = \min \text{Supp} \beta$.

PROPOSITION 5. *The map ν is a valuation on L , and is independent of the choice of S . The value group is G and the residue field is R .*

Proof. For $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, define

$$w(\alpha) = \min_{g \in G} \left\{ g + \nu' \left(\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n \right) \right\}.$$

The elements in the “min” belong to $(\text{Supp } \alpha) + \mathbf{N}) \cup \{\infty\}$, which is well-ordered by part 2 of Lemma 1, so this is well defined. It's clearly unchanged if an element of N is added to α . In particular, if we do so to get an element $\alpha' \in A((G))$ with coefficients in S , we find $w(\alpha) = w(\alpha') = \min \text{Supp } \alpha'$. Thus if β is the image of α in L , $\nu(\alpha) = w(\beta)$. Since w is independent of the choice of S , so is ν . If α', β' are the representatives in $A((G))$ with coefficients in S of elements $\alpha, \beta \in L$, then it is clear that $w(\alpha'\beta') = w(\alpha') + w(\beta')$ (because the leading coefficient of $\alpha'\beta'$ has valuation 0 under ν') and that $w(\alpha' + \beta') \geq \min\{w(\alpha'), w(\beta')\}$. Thus ν is a valuation.

The value group of ν is all of G , since $\nu(p^g) = g$ for any $g \in G$. The natural inclusion $A \subset A((G))$ composed with the quotient map $A((G)) \rightarrow L$ maps A into the valuation ring of L , which consists of series $\sum_{g \geq 0} \alpha_g p^g$, so it induces a map ϕ from A to the residue field of L . The residue class of $\sum_{g \geq 0} \alpha_g p^g$ equals $\phi(\alpha_0) \in A$ (since the maximal ideal for L consists of series $\sum_{g > 0} \alpha_g p^g$). Thus ϕ is surjective. Its kernel is the maximal ideal of A , so ϕ induces an isomorphism from the residue class field of A to that of L . \square

For example, if R is any perfect field of characteristic p , and $G = k^{-1}\mathbf{Z}$ for some $k \geq 1$ (with its copy of \mathbf{Z} as a subgroup of index k), then $L = R'(\sqrt[k]{p})$ with the p -adic valuation.

LEMMA 3. *If $\alpha = \sum_{g \in G} \alpha_g p^g$ and $\beta = \sum_{g \in G} \beta_g p^g$ with $\alpha_g, \beta_g \in S$ are two elements of L , then $\nu(\alpha - \beta) = \min\{g \in G \mid \alpha_g \neq \beta_g\}$. (The corresponding fact for the usual Mal'cev-Neumann fields is obvious.)*

Proof. Let w be the map used in the proof of the previous proposition. Let $\alpha' = \sum_{g \in G} \alpha_g t^g$ and $\beta' = \sum_{g \in G} \beta_g t^g$ in $A((G))$. Then $\nu(\alpha - \beta) = w(\alpha' - \beta')$. If $g_0 = \min\{g \in G \mid \alpha_g \neq \beta_g\}$, then the leading term of $\alpha' - \beta'$ is $(\alpha_{g_0} - \beta_{g_0})t^{g_0}$, and the leading coefficient here has valuation 0 under ν' , since $\alpha_{g_0}, \beta_{g_0}$ represent distinct residue classes, so $w(\alpha' - \beta') = g_0$, as desired. \square

Remarks. Since the construction of A from R is functorial (the Witt functor), it is clear that the construction of L from R is functorial as well (for

each G). However, whereas the Witt functor is fully faithful on perfect fields of characteristic p , this new functor is not. For example, Proposition 11 (to be proved in Section 7) shows L can have many continuous (i.e. valuation-preserving) automorphisms not arising from automorphisms of R .

Our construction could be done starting from a non-abelian value group to produce p -adic Mal'cev-Neumann division rings, but we will not be interested in such objects.

5. MAXIMALITY OF MAL'CEV-NEUMANN FIELDS

A valued field (E, w) is an *immediate extension* of another valued field (F, v) if

- (1) E is a field extension of F , and $w|_F = v$.
- (2) (E, w) and (F, v) have the same value groups and residue fields.

A valued field (F, v) is *maximally complete* if it has no immediate extensions other than (F, v) itself. (These definitions are due to F.K. Schmidt, but were first published by Krull [8].) For example, an easy argument shows that any field F with the trivial valuation, or with a discrete valuation making it complete, is maximally complete.

PROPOSITION 6. *Let (F, v) be a maximally complete valued field with value group G and residue field R . Then*

- (1) F is complete.
- (2) If R is algebraically closed and G is divisible, then F is algebraically closed.

Proof. (1) The completion \hat{F} of F is an immediate extension of F (see Proposition 5 in Chapter VI, §5, no. 3 of [2]), so $\hat{F} = F$.

(2) The algebraic closure \bar{F} of F is in this case an immediate extension of F (see Proposition 6 in Chapter VI, §3, no. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]), so $\bar{F} = F$.

(This delightful trick is due to MacLane [10].) \square

PROPOSITION 7. *Any continuous endomorphism of a maximally complete field F which induces the identity on the residue field is automatically an automorphism (i.e., surjective).*

Proof. The field F is an immediate extension of the image of the endomorphism, which is maximally complete since it's isomorphic to F . \square