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be embedded in a divisible one, namely its injective hull. Since an ordered group G is necessarily torsion-free, its injective hull  $\tilde{G}$  can be identified with the set of quotients g/m with  $g \in G$ , m a positive integer, modulo the equivalence relation  $g/m \sim h/n$  iff ng = mh in G. We make  $\tilde{G}$  an ordered group by setting  $g/m \ge h/n$  iff  $ng \ge mh$  in G. (One can check that this is the unique extension to  $\tilde{G}$  of the ordered group structure on G.)

If G is an ordered group, let  $G_{\infty} = G \cup \{\infty\}$  be the ordered monoid containing G in which  $g + \infty = \infty + g = \infty$  for all  $g \in G_{\infty}$  and  $g < \infty$  for all  $g \in G$ . As usual, a valuation v on a field F is a function from F to  $G_{\infty}$ satisfying for all  $x, y \in F$ 

(1) 
$$v(x) = \infty \text{ iff } x = 0$$

(2) 
$$v(xy) = v(x) + v(y)$$

(3)  $v(x+y) \ge \min \{v(x), v(y)\}.$ 

The value group is G. The valuation ring A is  $\{x \in F \mid v(x) \ge 0\}$ . This is a local ring with maximal ideal  $\mathcal{M} = \{x \in F \mid v(x) > 0\}$ . The residue field is  $A/\mathcal{M}$ . We refer to the pair (F, v) (or sometimes simply F) as a valued field.

## 3. MAL'CEV-NEUMANN RINGS

This section serves not only as review, but also as preparation for the construction of the next section. Mal'cev-Neumann rings are generalizations of Laurent series rings. For any ring R (all our rings are commutative with 1), and any ordered group G, the Mal'cev-Neumann ring R((G)) is defined as the set of formal sums  $\alpha = \sum_{g \in G} \alpha_g t^g$  in an indeterminate t with  $\alpha_g \in R$  such that the set  $\text{Supp } \alpha = \{g \in G \mid \alpha_g \neq 0\}$  is a well-ordered subset of G (under the given order of G). (Often authors suppress the indeterminate and write the sums in the form  $\sum \alpha_g g$ , as in a group ring. We use the indeterminate in order to make clear the analogy with the fields of the next section.) If  $\alpha = \sum_{g \in G} \alpha_g t^g$  and  $\beta = \sum_{g \in G} \beta_g t^g$  are elements of R((G)), then  $\alpha + \beta$  is defined as  $\sum_{g \in G} (\alpha_g + \beta_g) t^g$ , and  $\alpha\beta$  is defined by a "distributive law" as  $\sum_{j \in G} \gamma_j t^j$  where  $\gamma_j = \sum_{g + h = j} \alpha_g \beta_h$ .

LEMMA 1. Let A, B be well-ordered subsets of an ordered group G. Then

(1) If  $x \in G$ , then  $A \cap (-B+x)$  is finite. (We define  $-B + x = \{-b + x \mid b \in B\}$ .)

- (2) The set  $A + B = \{a + b \mid a \in A, b \in B\}$  is well-ordered.
- (3) The set  $A \cup B$  is well-ordered.

*Proof.* See [13].

The lemma above easily implies that the sum defining  $\gamma_j$  is always finite, and that Supp  $(\alpha + \beta)$  and Supp  $(\alpha\beta)$  are well-ordered. Once one knows that the operations are defined, it's clear that they make R((G)) a ring.

Define  $v: R((G)) \to G_{\infty}$  by  $v(0) = \infty$  and  $v(\alpha) = \min \text{Supp } \alpha$  for  $\alpha \neq 0$ . (This makes sense since Supp  $\alpha$  is well-ordered.) If  $\alpha \in R((G))$  is nonzero and  $v(\alpha) = g$ , we call  $\alpha_g t^g$  the *leading term* of  $\alpha$  and  $\alpha_g$  the *leading coefficient*. If R is a field, then v is a valuation on R((G)), since the leading term of  $\alpha$  approaches the product of the leading terms.

LEMMA 2. If  $\alpha \in R((G))$  satisfies  $v(\alpha) > 0$ , then  $1 - \alpha$  is a unit in R((G)).

**Proof.** One way of proving this is to show that for each  $g \in G$ , the coefficients of  $t^g$  in 1,  $\alpha$ ,  $\alpha^2$ , ... are eventually zero, so  $1 + \alpha + \alpha^2 + \cdots$  can be defined termwise. Then one needs to check that its support is well-ordered, and that it's an inverse for  $1 - \alpha$ . See [13] for this. An easier way [15] is to obtain an inverse of  $1 - \alpha$  by successive approximation.

COROLLARY 1. If the leading coefficient of  $\alpha \in R((G))$  is a unit of R, then  $\alpha$  is a unit of R((G)).

**Proof.** Let  $rt^g$  be the leading term of  $\alpha$ . Then  $\alpha$  is the product of  $rt^g$ , which is a unit in R((G)) with inverse  $r^{-1}t^{-g}$ , and  $(rt^g)^{-1}\alpha$ , which is a unit by the preceding lemma.

# COROLLARY 2. If R is a field, then R((G)) is a field.

So in this case, if we set K = R((G)), (K, v) is a valued field. Clearly the value group is all of G, and the residue field is R. Note that char K = char R, since in fact, R can be identified with a subfield of K. (We will refer to these fields as being the "equal characteristic" case, in contrast with the p-adic fields of the next section in which the fields have characteristic different from that of their residue fields.) For example, if  $G = \mathbb{Z}$ , then R((G)) is the usual field of formal Laurent series.