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be embedded in a divisible one, namely its injective hull. Since an ordered group G is necessarily torsion-free, its injective hull $\tilde{G}$ can be identified with the set of quotients g/m with $g \in G$, m a positive integer, modulo the equivalence relation $g/m \sim h/n$ iff $ng = mh$ in G. We make $\tilde{G}$ an ordered group by setting $g/m \geqslant h/n$ iff $ng \geqslant mh$ in G. (One can check that this is the unique extension to $\tilde{G}$ of the ordered group structure on G.)

If G is an ordered group, let $G_\infty = G \cup \{\infty\}$ be the ordered monoid containing G in which $g + \infty = \infty + g = \infty$ for all $g \in G_\infty$ and $g < \infty$ for all $g \in G$. As usual, a *valuation* v on a field F is a function from F to G_∞ satisfying for all $x, y \in F$

$$(1) \qquad v(x) = \infty \text{ iff } x = 0$$

$$(2) \qquad v(xy) = v(x) + v(y)$$

$$(3) \qquad v(x+y) \geqslant \min\{v(x), v(y)\}.$$

The *value group* is G. The *valuation ring* A is $\{x \in F \mid v(x) \geqslant 0\}$. This is a local ring with maximal ideal $\mathscr{M} = \{x \in F \mid v(x) > 0\}$. The *residue field* is $A/\mathscr{M}$. We refer to the pair (F, v) (or sometimes simply F) as a *valued field.*

3. MAL'CEV-NEUMANN RINGS

This section serves not only as review, but also as preparation for the construction of the next section. Mal'cev-Neumann rings are generalizations of Laurent series rings. For any ring R (all our rings are commutative with 1), and any ordered group G, the Mal'cev-Neumann ring $R((G))$ is defined as the set of formal sums $\alpha = \sum_{g \in G} \alpha_g t^g$ in an indeterminate t with $\alpha_g \in R$ such that the set $\operatorname{Supp} \alpha = \{g \in G \mid \alpha_g \neq 0\}$ is a well-ordered subset of G (under the given order of G). (Often authors suppress the indeterminate and write the sums in the form $\sum \alpha_g g$, as in a group ring. We use the indeterminate in order to make clear the analogy with the fields of the next section.) If $\alpha = \sum_{g \in G} \alpha_g t^g$ and $\beta = \sum_{g \in G} \beta_g t^g$ are elements of $R((G))$, then $\alpha + \beta$ is defined as $\sum_{g \in G} (\alpha_g + \beta_g) t^g$, and $\alpha\beta$ is defined by a "distributive law" as $\sum_{j \in G} \gamma_j t^j$ where $\gamma_j = \sum_{g+h=j} \alpha_g \beta_h$.

LEMMA 1. *Let A, B be well-ordered subsets of an ordered group G. Then*

(1) *If $x \in G$, then $A \cap (-B + x)$ is finite.*
(We define $-B + x = \{-b + x \mid b \in B\}$.)

(2) *The set* $A + B = \{a + b \mid a \in A, b \in B\}$ *is well-ordered.*

(3) *The set* $A \cup B$ *is well-ordered.*

Proof. See [13]. $\square$

The lemma above easily implies that the sum defining γ_j is always finite, and that $\operatorname{Supp}(\alpha + \beta)$ and $\operatorname{Supp}(\alpha\beta)$ are well-ordered. Once one knows that the operations are defined, it's clear that they make $R((G))$ a ring.

Define $\upsilon: R((G)) \to G_\infty$ by $\upsilon(0) = \infty$ and $\upsilon(\alpha) = \min \operatorname{Supp} \alpha$ for $\alpha \neq 0$. (This makes sense since $\operatorname{Supp} \alpha$ is well-ordered.) If $\alpha \in R((G))$ is nonzero and $\upsilon(\alpha) = g$, we call $\alpha_g t^g$ the *leading term* of α and α_g the *leading coefficient*. If R is a field, then υ is a valuation on $R((G))$, since the leading term of a product is the product of the leading terms.

LEMMA 2. *If* $\alpha \in R((G))$ *satisfies* $\upsilon(\alpha) > 0$, *then* $1 - \alpha$ *is a unit in* $R((G))$.

Proof. One way of proving this is to show that for each $g \in G$, the coefficients of t^g in $1, \alpha, \alpha^2, \ldots$ are eventually zero, so $1 + \alpha + \alpha^2 + \cdots$ can be defined termwise. Then one needs to check that its support is well-ordered, and that it's an inverse for $1 - \alpha$. See [13] for this. An easier way [15] is to obtain an inverse of $1 - \alpha$ by successive approximation. $\square$

COROLLARY 1. *If the leading coefficient of* $\alpha \in R((G))$ *is a unit of* R, *then* α *is a unit of* $R((G))$.

Proof. Let rt^g be the leading term of α. Then α is the product of rt^g, which is a unit in $R((G))$ with inverse $r^{-1}t^{-g}$, and $(rt^g)^{-1}\alpha$, which is a unit by the preceding lemma. $\square$

COROLLARY 2. *If* R *is a field, then* $R((G))$ *is a field.*

So in this case, if we set $K = R((G))$, (K, υ) is a valued field. Clearly the value group is all of G, and the residue field is R. Note that $\operatorname{char} K = \operatorname{char} R$, since in fact, R can be identified with a subfield of K. (We will refer to these fields as being the "equal characteristic" case, in contrast with the p-adic fields of the next section in which the fields have characteristic different from that of their residue fields.) For example, if $G = \mathbf{Z}$, then $R((G))$ is the usual field of formal Laurent series.