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Autor: Poonen, Bjorn

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(using the linearity of the Frobenius automorphism) that this series does formally satisfy our polynomial equation! (The other solutions are obtained by adding elements of $\mathbf{F}_p$ to this one.)

It is natural to seek a context in which series such as these make sense. If one tries to define a field containing all series $\sum_{q \in \mathbf{Q}} \alpha_q t^q$, one fails for the reason that multiplication is not well defined. But then one notices that a sequence of exponents coming from a transfinite successive approximation process must be well-ordered. If one considers only series in which the set of exponents is a well-ordered subset of $\mathbf{Q}$, one does indeed obtain a field.

Such fields are commonly known as Mal'cev-Neumann rings. (We will review their construction in Section 3.) They were introduced by Hahn in 1908, and studied in terms of valuations by Krull [8] in 1932. (Mal'cev [11] in 1948 and Neumann [12] in 1949 showed that the same construction could be performed for exponents in a non-abelian group to produce a division ring.)

If one tries to find p-adic expansions of elements algebraic over $\mathbf{Q}_p$, one encounters a similar situation. One is therefore led to construct p-adic analogues of the Mal'cev-Neumann rings. (See Section 4.) This construction is apparently new, except that Lampert [9] in 1986 described the special case of value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$ without giving details of a construction. (We will discuss this special case in detail in Section 7.)

In Section 5 we prove our main theorems. A corollary of our Theorem 2 is that a Mal'cev-Neumann ring (standard or p-adic) with divisible value group G and algebraically closed residue field R has the amazing property that every other valued field with the same value group, the same residue field, and the same restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$) can be embedded in the Mal'cev-Neumann ring. (We assume implicitly in the minimal subfield assumption that in the p-adic case the valuation of p must be the same element of G for the two fields.) Kaplansky [5] proved the existence of a field with this property using a different method. He also knew that it was a Mal'cev-Neumann ring when the restriction of the valuation to the minimal subfield is trivial, but was apparently unaware of its structure in the p-adic case.

2. Preliminaries

All ordered groups G in this paper are assumed to be abelian, and we write the group law additively. We call G *divisible* if for every $g \in G$ and positive integer n, the equation $nx = g$ has a solution in G. Every ordered group can be embedded in a divisible one, namely its injective hull. Since an ordered group G is necessarily torsion-free, its injective hull $\tilde{G}$ can be identified with the set of quotients g/m with $g \in G$, m a positive integer, modulo the equivalence relation $g/m \sim h/n$ iff $ng = mh$ in G. We make $\tilde{G}$ an ordered group by setting $g/m \geqslant h/n$ iff $ng \geqslant mh$ in G. (One can check that this is the unique extension to $\tilde{G}$ of the ordered group structure on G.)

If G is an ordered group, let $G_\infty = G \cup \{\infty\}$ be the ordered monoid containing G in which $g + \infty = \infty + g = \infty$ for all $g \in G_\infty$ and $g < \infty$ for all $g \in G$. As usual, a *valuation* v on a field F is a function from F to G_∞ satisfying for all $x, y \in F$

$$(1) \qquad v(x) = \infty \text{ iff } x = 0$$

$$(2) \qquad v(xy) = v(x) + v(y)$$

$$(3) \qquad v(x+y) \geqslant \min\{v(x), v(y)\}.$$

The *value group* is G. The *valuation ring* A is $\{x \in F \mid v(x) \geqslant 0\}$. This is a local ring with maximal ideal $\mathscr{M} = \{x \in F \mid v(x) > 0\}$. The *residue field* is $A/\mathscr{M}$. We refer to the pair (F, v) (or sometimes simply F) as a *valued field.*

3. MAL'CEV-NEUMANN RINGS

This section serves not only as review, but also as preparation for the construction of the next section. Mal'cev-Neumann rings are generalizations of Laurent series rings. For any ring R (all our rings are commutative with 1), and any ordered group G, the Mal'cev-Neumann ring $R((G))$ is defined as the set of formal sums $\alpha = \sum_{g \in G} \alpha_g t^g$ in an indeterminate t with $\alpha_g \in R$ such that the set $\operatorname{Supp} \alpha = \{g \in G \mid \alpha_g \neq 0\}$ is a well-ordered subset of G (under the given order of G). (Often authors suppress the indeterminate and write the sums in the form $\sum \alpha_g g$, as in a group ring. We use the indeterminate in order to make clear the analogy with the fields of the next section.) If $\alpha = \sum_{g \in G} \alpha_g t^g$ and $\beta = \sum_{g \in G} \beta_g t^g$ are elements of $R((G))$, then $\alpha + \beta$ is defined as $\sum_{g \in G} (\alpha_g + \beta_g) t^g$, and $\alpha\beta$ is defined by a "distributive law" as $\sum_{j \in G} \gamma_j t^j$ where $\gamma_j = \sum_{g+h=j} \alpha_g \beta_h$.

LEMMA 1. *Let A, B be well-ordered subsets of an ordered group G. Then*

(1) *If $x \in G$, then $A \cap (-B + x)$ is finite.*
(We define $-B + x = \{-b + x \mid b \in B\}$.)