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MAXIMALLY COMPLETE FIELDS

by Bjorn POONEN

ABSTRACT. Kaplansky proved in 1942 that among all fields with a valuation having a given divisible value group G, a given algebraically closed residue field R, and a given restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$), there is one that is maximal in the strong sense that every other can be embedded in it. In this paper, we construct this field explicitly and use the explicit form to give a new proof of Kaplansky's result. The field turns out to be a Mal'cev-Neumann ring or a p-adic version of a Mal'cev-Neumann ring in which the elements are formal series of the form $\sum_{g \in S} \alpha_g p^g$ where S is a well-ordered subset of G and the α_g's are residue class representatives. We conclude with some remarks on the p-adic Mal'cev-Neumann field containing $\bar{\mathbf{Q}}_p$.

I. INTRODUCTION

It is well known that if k is an algebraically closed field of characteristic zero, then the algebraic closure of the field of Laurent series $k((t))$ is obtained by adjoining $t^{1/n}$ for each integer $n \geqslant 1$, and that the expansion of a solution to a polynomial equation over $k((t))$ can be obtained by the method of successive approximation. (For example, to find a square root of $1 + t$, one solves for the coefficients of $1, t, t^2, \ldots$ in turn.) But if k is algebraically closed of characteristic p, $\cup_{n=1}^{\infty} k((t^{1/n}))$ is no longer an algebraic closure of $k((t))$. In particular, the Artin-Schreier equation $x^p - x = t^{-1}$ has no solution in $\cup_{n=1}^{\infty} k((t^{1/n}))$. (See p. 64 of Chevalley [3].) If one attempts nevertheless to successively approximate a solution, one obtains the expansion (due to Abhyankar [1])

$$x = t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \cdots,$$

in which the exponents do not tend to ∞, as they should if the series were to converge with respect to a valuation in the usual sense. However, one checks

(using the linearity of the Frobenius automorphism) that this series does formally satisfy our polynomial equation! (The other solutions are obtained by adding elements of $\mathbf{F}_p$ to this one.)

It is natural to seek a context in which series such as these make sense. If one tries to define a field containing all series $\sum_{q \in \mathbf{Q}} \alpha_q t^q$, one fails for the reason that multiplication is not well defined. But then one notices that a sequence of exponents coming from a transfinite successive approximation process must be well-ordered. If one considers only series in which the set of exponents is a well-ordered subset of $\mathbf{Q}$, one does indeed obtain a field.

Such fields are commonly known as Mal'cev-Neumann rings. (We will review their construction in Section 3.) They were introduced by Hahn in 1908, and studied in terms of valuations by Krull [8] in 1932. (Mal'cev [11] in 1948 and Neumann [12] in 1949 showed that the same construction could be performed for exponents in a non-abelian group to produce a division ring.)

If one tries to find p-adic expansions of elements algebraic over $\mathbf{Q}_p$, one encounters a similar situation. One is therefore led to construct p-adic analogues of the Mal'cev-Neumann rings. (See Section 4.) This construction is apparently new, except that Lampert [9] in 1986 described the special case of value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$ without giving details of a construction. (We will discuss this special case in detail in Section 7.)

In Section 5 we prove our main theorems. A corollary of our Theorem 2 is that a Mal'cev-Neumann ring (standard or p-adic) with divisible value group G and algebraically closed residue field R has the amazing property that every other valued field with the same value group, the same residue field, and the same restriction to the minimal subfield (either the trivial valuation on $\mathbf{Q}$ or $\mathbf{F}_p$, or the p-adic valuation on $\mathbf{Q}$) can be embedded in the Mal'cev-Neumann ring. (We assume implicitly in the minimal subfield assumption that in the p-adic case the valuation of p must be the same element of G for the two fields.) Kaplansky [5] proved the existence of a field with this property using a different method. He also knew that it was a Mal'cev-Neumann ring when the restriction of the valuation to the minimal subfield is trivial, but was apparently unaware of its structure in the p-adic case.

2. Preliminaries

All ordered groups G in this paper are assumed to be abelian, and we write the group law additively. We call G *divisible* if for every $g \in G$ and positive integer n, the equation $nx = g$ has a solution in G. Every ordered group can

be embedded in a divisible one, namely its injective hull. Since an ordered group G is necessarily torsion-free, its injective hull $\tilde{G}$ can be identified with the set of quotients g/m with $g \in G$, m a positive integer, modulo the equivalence relation $g/m \sim h/n$ iff $ng = mh$ in G. We make $\tilde{G}$ an ordered group by setting $g/m \geqslant h/n$ iff $ng \geqslant mh$ in G. (One can check that this is the unique extension to $\tilde{G}$ of the ordered group structure on G.)

If G is an ordered group, let $G_\infty = G \cup \{\infty\}$ be the ordered monoid containing G in which $g + \infty = \infty + g = \infty$ for all $g \in G_\infty$ and $g < \infty$ for all $g \in G$. As usual, a *valuation* v on a field F is a function from F to G_∞ satisfying for all $x, y \in F$

$$v(x) = \infty \text{ iff } x = 0 \tag{1}$$

$$v(xy) = v(x) + v(y) \tag{2}$$

$$v(x + y) \geqslant \min\{v(x), v(y)\}. \tag{3}$$

The *value group* is G. The *valuation ring* A is $\{x \in F \mid v(x) \geqslant 0\}$. This is a local ring with maximal ideal $\mathcal{M} = \{x \in F \mid v(x) > 0\}$. The *residue field* is $A/\mathcal{M}$. We refer to the pair (F, v) (or sometimes simply F) as a *valued field.*

3. Mal'cev-Neumann rings

This section serves not only as review, but also as preparation for the construction of the next section. Mal'cev-Neumann rings are generalizations of Laurent series rings. For any ring R (all our rings are commutative with 1), and any ordered group G, the Mal'cev-Neumann ring $R((G))$ is defined as the set of formal sums $\alpha = \sum_{g \in G} \alpha_g t^g$ in an indeterminate t with $\alpha_g \in R$ such that the set $\operatorname{Supp} \alpha = \{g \in G \mid \alpha_g \neq 0\}$ is a well-ordered subset of G (under the given order of G). (Often authors suppress the indeterminate and write the sums in the form $\sum \alpha_g g$, as in a group ring. We use the indeterminate in order to make clear the analogy with the fields of the next section.) If $\alpha = \sum_{g \in G} \alpha_g t^g$ and $\beta = \sum_{g \in G} \beta_g t^g$ are elements of $R((G))$, then $\alpha + \beta$ is defined as $\sum_{g \in G} (\alpha_g + \beta_g) t^g$, and $\alpha\beta$ is defined by a "distributive law" as $\sum_{j \in G} \gamma_j t^j$ where $\gamma_j = \sum_{g + h = j} \alpha_g \beta_h$.

Lemma 1. *Let A, B be well-ordered subsets of an ordered group G. Then*

(1) *If $x \in G$, then $A \cap (-B + x)$ is finite.*
(We define $-B + x = \{-b + x \mid b \in B\}$.)

(2) *The set* $A + B = \{a + b \mid a \in A, b \in B\}$ *is well-ordered.*

(3) *The set* $A \cup B$ *is well-ordered.*

Proof. See [13]. □

The lemma above easily implies that the sum defining γ_j is always finite, and that $\mathrm{Supp}(\alpha + \beta)$ and $\mathrm{Supp}(\alpha\beta)$ are well-ordered. Once one knows that the operations are defined, it's clear that they make $R((G))$ a ring.

Define $v: R((G)) \to G_\infty$ by $v(0) = \infty$ and $v(\alpha) = \min \mathrm{Supp}\, \alpha$ for $\alpha \neq 0$. (This makes sense since $\mathrm{Supp}\, \alpha$ is well-ordered.) If $\alpha \in R((G))$ is nonzero and $v(\alpha) = g$, we call $\alpha_g t^g$ the *leading term* of α and α_g the *leading coefficient*. If R is a field, then v is a valuation on $R((G))$, since the leading term of a product is the product of the leading terms.

LEMMA 2. *If* $\alpha \in R((G))$ *satisfies* $v(\alpha) > 0$, *then* $1 - \alpha$ *is a unit in* $R((G))$.

Proof. One way of proving this is to show that for each $g \in G$, the coefficients of t^g in $1, \alpha, \alpha^2, \ldots$ are eventually zero, so $1 + \alpha + \alpha^2 + \cdots$ can be defined termwise. Then one needs to check that its support is well-ordered, and that it's an inverse for $1 - \alpha$. See [13] for this. An easier way [15] is to obtain an inverse of $1 - \alpha$ by successive approximation. □

COROLLARY 1. *If the leading coefficient of* $\alpha \in R((G))$ *is a unit of* R, *then* α *is a unit of* $R((G))$.

Proof. Let rt^g be the leading term of α. Then α is the product of rt^g, which is a unit in $R((G))$ with inverse $r^{-1}t^{-g}$, and $(rt^g)^{-1}\alpha$, which is a unit by the preceding lemma. □

COROLLARY 2. *If* R *is a field, then* $R((G))$ *is a field.*

So in this case, if we set $K = R((G))$, (K, v) is a valued field. Clearly the value group is all of G, and the residue field is R. Note that $\mathrm{char}\, K = \mathrm{char}\, R$, since in fact, R can be identified with a subfield of K. (We will refer to these fields as being the "equal characteristic" case, in contrast with the p-adic fields of the next section in which the fields have characteristic different from that of their residue fields.) For example, if $G = \mathbf{Z}$, then $R((G))$ is the usual field of formal Laurent series.

4. p-ADIC MAL'CEV-NEUMANN FIELDS

To construct analogous examples of characteristic zero whose residue field has nonzero characteristic requires a more complicated construction. First we recall two results about complete discrete valuation rings. For proofs, see [17], pp. 32-34.

A valued field (F, v) is called *discrete* if $v(F) = \mathbf{Z}$.

PROPOSITION 1. *If R is a perfect field of characteristic $p > 0$, then there exists a unique field R' of characteristic 0 with a discrete valuation v such that the residue field is R, $v(p) = 1 \in \mathbf{Z}$, and R' is complete with respect to v. (The valuation ring A of R' is called the ring of* Witt vectors *with coefficients in R.)*

For example, if $R = \mathbf{F}_p$, then $R' = \mathbf{Q}_p$ with the p-adic valuation.

PROPOSITION 2. *Suppose F is field with a discrete valuation v, and $t \in F$ satisfies $v(t) = 1$. Let $S \subset F$ be a set of representatives for the residue classes with $0 \in S$. Then every element $x \in F$ can be written uniquely as $\sum_{m \in \mathbf{Z}} x_m t^m$, where $x_m \in S$ for each m, and $x_m = 0$ for all sufficiently negative m. Conversely, if F is complete, every such series defines an element of F.*

Now for the construction. Let R be a perfect field of characteristic p, and let G be an ordered group containing $\mathbf{Z}$ as a subgroup, or equivalently with a distinguished positive element. (When we eventually define our valuation v, this element $1 \in G$ will be $v(p)$.) Let A be the valuation ring of the valued field (R', v') given by Proposition 1.

What we want is to have the indeterminate t stand for p in elements of $A((G))$, so we get elements of the form $\sum_{g \in G} \alpha_g p^g$. The problem is that some elements of $A((G))$, like $-p + t^1$, "should be" zero. So what we do is to take a quotient $A((G))/N$ where $N \subset A((G))$ is the ideal of elements that "should be" zero.

We say that $\alpha = \sum_g \alpha_g t^g \in A((G))$ is a *null series* if for all $g \in G$, $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = 0$ in R'. (Recall that we fixed a copy of $\mathbf{Z}$ in G.) Note that $\alpha_{g+n} = 0$ for sufficiently negative n, since otherwise $\operatorname{Supp} \alpha$ would not be well-ordered. Also, $v'(\alpha_{g+n} p^n) \geqslant n$, so $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n$ always converges in R'. Let N be the set of null series.

PROPOSITION 3. *N is an ideal of $A((G))$.*

Proof. Clearly N is an additive subgroup. Let $G' \subset G$ be a set of coset representatives for $G/\mathbf{Z}$. Suppose $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, $\beta = \sum_{h \in G} \beta_h t^h \in N$, and $\alpha\beta = \sum_{j \in G} \gamma_j t^j$. Then for each $j \in G$,

$$\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = \sum_{\substack{g+h=j+n \\ n \in \mathbf{Z}}} \alpha_g \beta_h p^n$$

$$= \sum_{\substack{h' \in G' \\ l, m \in \mathbf{Z}}} (\alpha_{j-h'+l} p^l)(\beta_{h'+m} p^m)$$

(We write $h = h' + m$ with $h' \in G'$ and let $l = n - m$.)

Since $\beta \in N$, $\sum_{m \in \mathbf{Z}} \beta_{h'+m} p^m = 0$ for each $h' \in G$, so we get $\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = 0$. (These infinite series manipulations in R' are valid, because for each $i \in \mathbf{Z}$, only finitely many terms have valuation less than i, since each γ_{j+n} is a finite sum of products $\alpha_g \beta_h$.) Hence N is an ideal. □

Define the p-adic Mal'cev-Neumann field L as $A((G))/N$.

PROPOSITION 4. *Let $S \subset A$ be a set of representatives for the residue classes of A, with $0 \in S$. Then any element $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$ is equivalent modulo N to a element $\beta = \sum_{g \in G} \beta_g t^g$ with each β_g in S. Moreover, β is unique.*

Proof. Let $G' \subset G$ be a set of coset representatives for $G/\mathbf{Z}$. For each $g \in G'$, we may write

$$\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = \sum_{n \in \mathbf{Z}} \beta_{g+n} p^n$$

with $\beta_{g+n} \in S$, by Proposition 2. (This is possible since R' is complete with respect to its discrete valuation.) Then $\beta = \sum_{g \in G'} \sum_{n \in \mathbf{Z}} \beta_{g+n} t^n$ is a well-defined element of $A((G))$, since $\operatorname{Supp}(\beta) \subseteq (\operatorname{Supp} \alpha) + \mathbf{N}$, which is well-ordered by part 2 of Lemma 1. Finally $\alpha - \beta \in N$, by definition of N. The uniqueness follows from the uniqueness in Proposition 2. □

COROLLARY 3. *$L = A((G))/N$ is a field.*

Proof. The previous proposition shows that any $\alpha \in A((G))$ is equivalent modulo N to 0 or an element which is a unit in $A((G))$ by Corollary 1. □

Proposition 4 allows us to write an element of L uniquely (and somewhat carelessly) as $\beta = \sum_{g \in G} \beta_g p^g$, with $\beta_g \in S$. Thus given S, we can speak of $\operatorname{Supp}(\beta)$ for $\beta \in L$. Define $v: L \to G_\infty$ by $v(\beta) = \min \operatorname{Supp} \beta$.

PROPOSITION 5. *The map* v *is a valuation on* L, *and is independent of the choice of* S. *The value group is* G *and the residue field is* R.

Proof. For $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, define

$$w(\alpha) = \min_{g \in G} \left\{ g + v'\left(\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n \right) \right\} .$$

The elements in the "min" belong to $(\mathrm{Supp}(\alpha) + \mathbf{N}) \cup \{\infty\}$, which is well-ordered by part 2 of Lemma 1, so this is well defined. It's clearly unchanged if an element of N is added to α. In particular, if we do so to get an element $\alpha' \in A((G))$ with coefficients in S, we find $w(\alpha) = w(\alpha') = \min \mathrm{Supp}\, \alpha'$. Thus if β is the image of α in L, $v(\alpha) = w(\beta)$. Since w is independent of the choice of S, so is v. If α', β' are the representatives in $A((G))$ with coefficients in S of elements $\alpha, \beta \in L$, then it is clear that $w(\alpha'\beta') = w(\alpha') + w(\beta')$ (because the leading coefficient of $\alpha'\beta'$ has valuation 0 under v') and that $w(\alpha' + \beta') \geqslant \min\{w(\alpha'), w(\beta')\}$. Thus v is a valuation.

The value group of v is all of G, since $v(p^g) = g$ for any $g \in G$. The natural inclusion $A \subset A((G))$ composed with the quotient map $A((G)) \to L$ maps A into the valuation ring of L, which consists of series $\sum_{g \geqslant 0} \alpha_g p^g$, so it induces a map ϕ from A to the residue field of L. The residue class of $\sum_{g \geqslant 0} \alpha_g p^g$ equals $\phi(\alpha_0) \in A$ (since the maximal ideal for L consists of series $\sum_{g > 0} \alpha_g p^g$). Thus ϕ is surjective. Its kernel is the maximal ideal of A, so ϕ induces an isomorphism from the residue class field of A to that of L. □

For example, if R is any perfect field of characteristic p, and $G = k^{-1}\mathbf{Z}$ for some $k \geqslant 1$ (with its copy of $\mathbf{Z}$ as a subgroup of index k), then $L = R'(\sqrt[k]{p})$ with the p-adic valuation.

LEMMA 3. *If* $\alpha = \sum_{g \in G} \alpha_g p^g$ *and* $\beta = \sum_{g \in G} \beta_g p^g$ *with* $\alpha_g, \beta_g \in S$ *are two elements of* L, *then* $v(\alpha - \beta) = \min\{g \in G \mid \alpha_g \neq \beta_g\}$. *(The corresponding fact for the usual Mal'cev-Neumann fields is obvious.)*

Proof. Let w be the map used in the proof of the previous proposition. Let $\alpha' = \sum_{g \in G} \alpha_g t^g$ and $\beta' = \sum_{g \in G} \beta_g t^g$ in $A((G))$. Then $v(\alpha - \beta) = w(\alpha' - \beta')$. If $g_0 = \min\{g \in G \mid \alpha_g \neq \beta_g\}$, then the leading term of $\alpha' - \beta'$ is $(\alpha_{g_0} - \beta_{g_0}) t^{g_0}$, and the leading coefficient here has valuation 0 under v', since $\alpha_{g_0}, \beta_{g_0}$ represent distinct residue classes, so $w(\alpha' - \beta') = g_0$, as desired. □

Remarks. Since the construction of A from R is functorial (the Witt functor), it is clear that the construction of L from R is functorial as well (for

each G). However, whereas the Witt functor is fully faithful on perfect fields of characteristic p, this new functor is not. For example, Proposition 11 (to be proved in Section 7) shows L can have many continuous (i.e. valuation-preserving) automorphisms not arising from automorphisms of R.

Our construction could be done starting from a non-abelian value group to produce p-adic Mal'cev-Neumann division rings, but we will not be interested in such objects.

5. Maximality of Mal'cev-Neumann fields

A valued field (E, w) is an *immediate extension* of another valued field (F, v) if

(1) E is a field extension of F, and $w|_F = v$.

(2) (E, w) and (F, v) have the same value groups and residue fields.

A valued field (F, v) is *maximally complete* if it has no immediate extensions other than (F, v) itself. (These definitions are due to F.K. Schmidt, but were first published by Krull [8].) For example, an easy argument shows that any field F with the trivial valuation, or with a discrete valuation making it complete, is maximally complete.

Proposition 6. *Let (F, v) be a maximally complete valued field with value group G and residue field R. Then*

(1) *F is complete.*

(2) *If R is algebraically closed and G is divisible, then F is algebraically closed.*

Proof. (1) The completion $\hat{F}$ of F is an immediate extension of F (see Proposition 5 in Chapter VI, §5, no. 3 of [2]), so $\hat{F} = F$.

(2) The algebraic closure $\bar{F}$ of F is in this case an immediate extension of F (see Proposition 6 in Chapter VI, §3, no. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]), so $\bar{F} = F$.

(This delightful trick is due to MacLane [10].) □

Proposition 7. *Any continuous endomorphism of a maximally complete field F which induces the identity on the residue field is automatically an automorphism (i.e., surjective).*

Proof. The field F is an immediate extension of the image of the endomorphism, which is maximally complete since it's isomorphic to F. □

From now on, when we refer to *Mal'cev-Neumann fields,* we mean one of the two fields K or L from the previous two sections. Let these have valuation v with value group G and residue field R. From now on, the proofs for the equal characteristic case K will be the same as (or easier than) those for the p-adic case L, so we will only give proofs for L. (To get a proof for K, simply replace p^g with t^g, and replace the set S of representatives with R.)

We will use the following lemma to show K and L are maximally complete.

LEMMA 4. *Let* (F, v) *be a valued field with value group* G. *Suppose we have an arbitrary system of inequalities of the form* $v(x - a_\sigma) \geqslant g_\sigma$, *with* $a_\sigma \in F$ *and* $g_\sigma \in G$ *for all* σ *in some index set* I. *Then*

(1) *If the system has a solution* $x \in F$, *then* $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ *for all* $\sigma_1, \sigma_2 \in I$.

(2) *Suppose in addition that* $F = L$ *(or K) is one of the Mal'cev-Neumann fields. Then the converse is true; i.e., if* $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ *for all* $\sigma_1, \sigma_2 \in I$, *then the system has a solution.*

Proof. (1) This is simply a consequence of the triangle inequality.

(2) Suppose $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all $\sigma_1, \sigma_2 \in I$. For each $g \in G$, let x_g be the coefficient of p^g in a_σ for any σ for which $g_\sigma > g$, and let $x_g = 0$ if no such σ exists. We claim x_g is uniquely defined. For if $g_{\sigma_1}, g_{\sigma_2} > g$, then $v(a_{\sigma_1} - a_{\sigma_2}) > g$, so by Lemma 3 the coefficients of p^g in $a_{\sigma_1}, a_{\sigma_2}$ must be the same.

Define $x = \sum_{g \in G} x_g p^g$. To show $x \in L$, we must check that $\operatorname{Supp} x$ is well-ordered. Suppose $h_1, h_2, \ldots$ is a strictly descending sequence within $\operatorname{Supp} x$. Then by definition of x_g, $h_1 < g_\sigma$ for some $\sigma \in I$, and $h_n \in \operatorname{Supp} a_\sigma$ for all $n \geqslant 1$. This is a contradiction, since $\operatorname{Supp} a_\sigma$ is well-ordered. Thus $x \in L$.

By definition of x_g, the coefficients of p^g in x and a_σ agree for $g < g_\sigma$. From Lemma 3 it follows that $v(x - a_\sigma) \geqslant g_\sigma$. □

THEOREM 1 (Krull [8]). *The Mal'cev-Neumann fields* K *and* L *are maximally complete. (Actually, Krull proved this only for the equal characteristic case* (K), *but his proof applies equally well to the p-adic fields* L.*)*

Proof. (As usual, we treat only the p-adic case.) Suppose (M, w) is a proper immediate extension of (L, v). Fix $\mu \in M \setminus L$. Consider the system of inequalities $w(x - a_\sigma) \geqslant g_\sigma$, where a_σ ranges over all elements of L and $g_\sigma = w(\mu - a_\sigma)$. Obviously μ is a solution (in M), so by part 1 of Lemma 4,

$w(a_{\sigma_1} - a_{\sigma_2}) \geq \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all σ_1, σ_2. Now $v(a_{\sigma_1} - a_{\sigma_2}) = w(a_{\sigma_1} - a_{\sigma_2}) \geq \min\{g_{\sigma_1}, g_{\sigma_2}\}$, so we may apply part 2 of Lemma 4 to deduce that the system of inequalities $v(x - a_\sigma) \geq g_\sigma$ has a solution $\lambda \in L$.

The idea is that λ is a best approximation in L to μ. We will contradict this by adding the "leading term" of the difference $\mu - \lambda$ to λ to get a better one. Since $\mu \notin L$, $\mu - \lambda \neq 0$, so we can let $g = w(\mu - \lambda) \in G$. (Here we are using that L and M have the same value group.) Then $w(p^{-g}(\mu - \lambda)) = 0$, so there exists a unique representative $s \in S$ for the (nonzero) residue class containing $p^{-g}(\mu - \lambda)$. (Here we are using that L and M have the same residue field.) Then $w(p^{-g}(\mu - \lambda) - s) > 0$, so $w(\mu - \lambda - sp^g) > g$. On the other hand, $g = v(-sp^g) = v(\lambda - (\lambda + sp^g)) \geq w(\mu - (\lambda + sp^g))$, by the definition of λ, using $a_\sigma = \lambda + sp^g$. This contradiction proves L is maximally complete. □

Remark. It is true in general that F is maximally complete iff part 2 of Lemma 4 is true for F. See Kaplansky's discussion of pseudolimits [5], and Theorem 5 in Chapter I of [4].

COROLLARY 4. *Any Mal'cev-Neumann field is complete. A Mal'cev-Neumann field with divisible value group and algebraically closed residue field is itself algebraically closed.*

Proof. Combine the previous theorem with Proposition 6. □

Remark. In practice, to find solutions to a polynomial equation over a Mal'cev-Neumann field, one can use successive approximation. This method could be used to give another (much messier) proof that these Mal'cev-Neumann fields are algebraically closed.

We will show that the Mal'cev-Neumann fields K and L are maximal in a sense much stronger than Theorem 1 implies. This will be made precise in Corollary 5.

THEOREM 2. *Suppose L (or K) is a Mal'cev-Neumann field with valuation v having divisible value group G and algebraically closed residue field R. Suppose E is a subfield of L, and that (F, w) is a valued field extension of (E, v), with value group contained in G and residue field contained in R. Then there exists an embedding of valued fields $\phi: F \to L$ which extends the inclusion $E \hookrightarrow L$.*

Proof. Since G is divisible and R is algebraically closed, we can extend the valuation on F to a valuation on $\bar{F}$ with value group in G and residue field in R, by Proposition 6 in Chapter VI, §3, No. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]. If we could find an embedding of $\bar{F}$ into L, we would get an embedding of F into L. Thus we may assume that F is algebraically closed.

Let $\mathscr{C}$ be the collection of pairs (E', ϕ) such that E' is a field between E and F and $\phi: E' \to L$ is an embedding of valued fields. Define a partial order on $\mathscr{C}$ by saying (E_2', ϕ_2) is above (E_1', ϕ_1) if $E_2' \supseteq E_1'$ and ϕ_2 extends ϕ_1. By Zorn's Lemma, we can find a maximal element (E', ϕ) of $\mathscr{C}$. By relabeling elements, we can assume $E' \subseteq L$, and we may as well rename E' as E.

We claim this E is algebraically closed. Both F and L are algebraically closed. (For L, this follows from Corollary 4.) So we have an algebraic closure of E in F and in L, each with a valuation extending the valuation on E. By Corollary 1 in Chapter VI, §8, No. 6 of [2], two such valuations can differ only by an automorphism of $\bar{E}$ over E; i.e., there exists a continuous embedding of the algebraic closure of E in F into L. By maximality of (E, ϕ) in $\mathscr{C}$, E must be algebraically closed already.

If $E = F$, we are done, so assume there is some element $\mu \in F \backslash E$. We will define a corresponding element $\mu' \in L$.

Case 1: There exists a best approximation $e_0 \in E$ to μ; i.e. there exists $e_0 \in E$ such that $w(\mu - e) \leqslant w(\mu - e_0)$ for all $e \in E$. Let $g = w(\mu - e_0) \in G$.

Case 1a: $g \notin v(E)$. Then define $\mu' = e_0 + p^g$.

Case 1b: $g = v(\delta)$ for some $\delta \in E$. Then $w(\delta^{-1}(\mu - e_0)) = 0$, so we let $s \in S$ be the representative of the (nonzero) residue class corresponding to $\delta^{-1}(\mu - e_0) \in F$, and define $\mu' = e_0 + s\delta$.

Note that in these cases, $v(\mu' - e_0) = g$, so for all $e \in E$,

$$\begin{aligned} v(\mu' - e) &\geqslant \min\{v(\mu' - e_0), v(e - e_0)\} \quad \text{(the triangle inequality)} \\ &= \min\{g, v(e - e_0)\} \\ &= \min\{w(\mu - e_0), w(e - e_0)\} \quad \text{(since } v \text{ and } w \text{ agree on } E\text{)} \\ &\geqslant \min\{w(\mu - e_0), w(\mu - e), w(\mu - e_0)\} \quad \text{(the triangle inequality)} \\ &= w(\mu - e) \quad \text{(by definition of } e_0\text{)}. \end{aligned}$$

Case 2: For every $e \in E$, there exists $e' \in E$ with $w(\mu - e') > w(\mu - e)$.

Consider the system of inequalities $w(x - e_\sigma) \geqslant g_\sigma$, where e_σ ranges over all elements of E and $g_\sigma = w(\mu - e_\sigma)$. Since μ is a solution (in F), $w(e_{\sigma_1} - e_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ by part 1 of Lemma 4. We have

$$v(e_{\sigma_1} - e_{\sigma_2}) = w(e_{\sigma_1} - e_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\},$$

so by part 2 of Lemma 4, the system of inequalities $v(x - e_\sigma) \geqslant g_\sigma$ has a solution μ' in L.

Claim. In all cases, $w(\mu - e) = v(\mu' - e)$ for all $e \in E$.

Proof. From the remarks at the end of Case 1, and by the definition of μ' in Case 2, we have $w(\mu - e) \leqslant v(\mu' - e)$ for all $e \in E$.

First suppose e is not a best approximation to μ, so $w(\mu - e') > w(\mu - e)$, for some $e' \in E$. Then equality holds in the triangle inequality,

$$w(e - e') = w((\mu - e') - (\mu - e)) = w(\mu - e)$$

so

$$v(e - e') = w(e - e') = w(\mu - e) < w(\mu - e') \leqslant v(\mu' - e') .$$

Again equality holds in the triangle inequality, so we get

$$v(\mu' - e) = v((\mu' - e') - (e - e')) = v(e - e') = w(\mu - e)$$

which proves the claim in this case.

Thus we are left with the case in which $w(\mu - e') \leqslant w(\mu - e)$ for all $e' \in E$. Then Case 1 holds and $w(\mu - e) = w(\mu - e_0) = g$. Suppose $v(\mu' - e) > g$. Then applying the triangle equality to $e - e_0 = (\mu' - e_0) - (\mu' - e)$ and using $v(\mu' - e_0)$ from our remarks at the end of Case 1, we get $v(e - e_0) = v(\mu' - e_0) = g$. Thus $g \in v(E)$ so we must be in Case 1b. Moreover

$$v(\delta^{-1}(\mu' - e_0) - \delta^{-1}(e - e_0)) = v(\delta^{-1}) + v(\mu' - e) > -g + g = 0$$

so $\delta^{-1}(\mu' - e_0)$ and $\delta^{-1}(e - e_0)$ have the same image in the residue field R. But by definition of μ' in Case 1b, $\delta^{-1}(\mu' - e_0)$ has the same image in R as $\delta^{-1}(\mu - e_0)$. Combining these facts gives us

$$w(\delta^{-1}(\mu - e_0) - \delta^{-1}(e - e_0)) > 0$$

so $w(\mu - e) > w(\delta) = v(\delta) = g$, contradicting the definitions of g and e_0. Thus we cannot have $v(\mu' - e) > g$. But we know $v(\mu' - e) \geqslant w(\mu - e) = g$, so we must have $v(\mu' - e) = w(\mu - e) = g$. This completes the proof of the claim.

Since $\mu \notin E$, $v(\mu' - e) = w(\mu - e) \neq \infty$ for all $e \in E$. Hence $\mu' \notin E$. But E is algebraically closed, so μ and μ' are transcendental over E, and we have an isomorphism of fields $\Phi: E(\mu) \to E(\mu')$ over E which maps μ to μ'.

We claim that Φ preserves the valuation. (The valuations on $E(\mu), E(\mu')$ are the restrictions of w, v respectively). Since E is algebraically closed , any element $\rho \in E(\mu)$ can be written

$$\rho = \varepsilon_0(\mu - \varepsilon_1)^{n_1}(\mu - \varepsilon_2)^{n_2} \cdots (\mu - \varepsilon_k)^{n_k} ,$$

for some $\varepsilon \in E$ and $n_i \in \mathbf{Z}$. By the Claim above, and the fact that v and w agree on E, it follows that $w(\rho) = v(\Phi(\rho))$, as desired.

But $(E(\mu), \Phi)$ contradicts the maximality of (E, ϕ) in $\mathscr{C}$. Thus we must have had $E = F$, so we are done. □

COROLLARY 5. *Let (F, v) be a valued field with value group contained in a divisible ordered group G, and residue field contained in an algebraically closed field R. Define K and L as usual as the Mal'cev-Neumann fields with value group G and residue field R. (Define the p-adic Mal'cev-Neumann field L only if $\operatorname{char} R > 0$.) Then there exists an embedding of valued fields $\phi: F \to K$ or $\phi: F \to L$, depending on if the restriction of v to the minimal subfield of F is the trivial valuation (on $\mathbf{Q}$ or $\mathbf{F}_p$) or the p-adic valuation on $\mathbf{Q}$.*

Proof. Apply Theorem 2 with E as the minimal subfield. □

COROLLARY 6. *Every valued field F has at least one immediate extension which is maximally complete. If the value group G is divisible and the residue field R is algebraically closed, then there is only one (up to isomorphism).*

Proof. Embed F in a Mal'cev-Neumann field L (or K) with value group $\tilde{G}$ and residue field $\bar{R}$, according to the previous corollary. Let $\mathscr{C}$ be the collection of valued subfields of L which are immediate extensions of F. By Zorn's Lemma, $\mathscr{C}$ has a maximal element M. If M had an immediate extension M', then by Theorem 2, we could embed M' in L. This would contradict the maximality of M.

If G is divisible and the R is algebraically closed, then any maximally complete immediate extension M of F can be embedded in L, and L is an immediate extension of M, so $L = M$. □

Remarks. Krull [8] was the first to prove that every valued field F had a maximal extension. His proof involves showing directly that there exists a bound on the cardinality of a valued field with given value group and residue field. Then Zorn's Lemma is applied.

Kaplansky [5] has investigated in detail the question of when the maximally complete immediate extension is unique. He has found weaker conditions on the value group and residue field which guarantee this extension is unique. If $\operatorname{char} R = 0$, the extension is unique. If $\operatorname{char} R = p > 0$, the extension is unique if the following pair of conditions is satisfied:

(1) Any equation of the form

$$x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x + a_{n+1} = 0$$

with coefficients in R has a root in R.

(2) The value group G satisfies $G = pG$.

Also if G is discrete of arbitrary rank and char F = char R, then the extension in unique [6]. But Kaplansky gives examples where the extension is not unique. The exact conditions under which the extension is unique are not known.

6. Applications

One application of Theorem 2 is to the problem of "glueing" two valued fields. (This result can also be proved directly without the use of Mal'cev-Neumann fields; it is equivalent to Exercise 2 for §2 in Chapter VI of [2]. Our method has the advantage of showing that the value group of the composite field can be contained in any divisible value group large enough to contain the value groups of the fields to be glued.)

Proposition 8. *Suppose E, F, F' are valued fields and that we are given embeddings of valued fields $\phi: E \to F, \phi': E \to F'$. Then there exist a Mal'cev-Neumann field L (or K) and embeddings of valued fields $\Phi: F \to L, \Phi': F' \to L$ such that $\Phi \circ \phi = \Phi' \circ \phi'$.*

Proof. By the glueing theorem for ordered groups [14], we can assume the value groups of F and F' are contained in a single ordered group G. Also we can assume that their residue fields are contained in a field R. Moreover, we may assume G is divisible and R is algebraically closed. Then E can be embedded as a valued subfield of a power series field L (or K) with value group G and residue field R, by Corollary 5. Finally, Theorem 2 gives us the desired embeddings Φ, Φ'. □

Remark. Transfinite induction can be used to prove the analogous result for glueing an arbitrary collection of valued fields.

Since a non-archimedean absolute value on a field can be interpreted as a valuation with value group contained in **R**, we can specialize the results of Section 5 to get results about fields with non-archimedean absolute values. For example, Corollary 5 implies the following, which may be considered the non-archimedean analogue of Ostrowski's theorem that any field with an archimedean absolute value can be embedded in **C** with its usual absolute value (or one equivalent).

PROPOSITION 9. *Let* $(F, |\,|)$ *be a field with a non-archimedean absolute value, and suppose the residue field is contained in the algebraically closed field* R. *Define* K *and* L *as the Mal'cev-Neumann fields with value group* $\mathbf{R}$ *and residue field* R. *(Define the p-adic Mal'cev-Neumann field* L *only if* $\operatorname{char} R > 0$.*) The valuations on* K *and* L *induce corresponding absolute values. Then there exists an absolute value-preserving embedding of fields* $\phi: F \to K$ *or* $\phi: F \to L$, *depending on if the restriction of* $|\,|$ *to the minimal subfield of* F *is the trivial absolute value (on* $\mathbf{Q}$ *or* $\mathbf{F}_p$*) or the p-adic absolute value on* $\mathbf{Q}$.

Similarly, Proposition 8 above gives a glueing proposition for non-archimedean absolute values. In fact, this result holds for archimedean absolute values as well, in light of Ostrowski's theorem.

7. EXAMPLE: THE MAXIMALLY COMPLETE IMMEDIATE EXTENSION OF $\bar{\mathbf{Q}}_p$

For this section, (L, v) will denote the p-adic Mal'cev-Neumann field having value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$. We have a natural embedding of $\mathbf{Q}_p$ into L. By Corollary 4, L is algebraically closed, so this embedding extends to an embedding of $\bar{\mathbf{Q}}_p$ into L (which is unique up to automorphisms of $\bar{\mathbf{Q}}_p$ over $\mathbf{Q}_p$.) In fact this embedding is continuous, since there is a unique valuation on $\bar{\mathbf{Q}}_p$ extending the p-adic valuation on $\mathbf{Q}_p$. Since $\bar{\mathbf{Q}}_p$ has value group $\mathbf{Q}$ and residue field $\bar{\mathbf{F}}_p$, L is an immediate extension of $\bar{\mathbf{Q}}_p$. By Corollary 6, L is in fact the unique maximally complete immediate extension of $\bar{\mathbf{Q}}_p$. Also, any valued field (F, w) of characteristic 0 satisfying

(1) The restriction of w to $\mathbf{Q}$ is the p-adic valuation;

(2) The value group is contained in $\mathbf{Q}$;

(3) The residue field is contained in $\bar{\mathbf{F}}_p$;

can be embedded in L, by Corollary 5. For example, the completion $\mathbf{C}_p$ of $\bar{\mathbf{Q}}_p$ can be embedded in L. (This could also be proved by noting that L is complete by Corollary 4.)

We will always use as the set S of representatives for $\bar{\mathbf{F}}_p$ the primitive k^{th} roots of 1, for all k not divisible by p, and 0. Then the elements of L have the form $\sum_g \alpha_g p^g$ for some primitive k^{th} roots α_g of 1, where the exponents form a well-ordered subset of $\mathbf{Q}$. In particular, the elements of $\bar{\mathbf{Q}}_p$ can be expressed in this form. This was first discovered by Lampert [9].

Example: (similar to those in [9]) Let p be an odd prime. The p^{th} roots of $1 - p$ in $\bar{\mathbf{Q}}_p$ have the expansion

$$1 - p^{1/p} + p^{1/p+1/p^2} - p^{1/p+1/p^2+1/p^3} + \cdots$$
$$+ \zeta p^{1/(p-1)} + (\text{higher order terms}),$$

where ζ is any one of the p solutions to $\zeta^p = -\zeta$ in $\bar{\mathbf{Q}}_p$.

PROPOSITION 10. *The fields L and $\mathbf{Q}_p$ have cardinality $2^{\aleph_0}$ (and hence so do all intermediate fields).*

Proof. Each series in L defines a distinct function $\mathbf{Q} \to \bar{\mathbf{F}}_p$ by sending q to the residue class of the coefficient of p^q. The number of such functions is $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, so $|L| \leqslant 2^{\aleph_0}$. On the other hand, as is well known, $|\mathbf{Q}_p| = 2^{\aleph_0}$ already, so the result follows. □

Since L and $\mathbf{C}_p$ are both complete algebraically closed fields of cardinality $2^{\aleph_0}$, it is natural to ask if $L = \mathbf{C}_p$. That L strictly contains $\mathbf{C}_p$ follows from Lampert's remark that the support of the series of an element of $\bar{\mathbf{Q}}_p$ is contained in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N, and that the residue classes of the coefficients in the series lie in $\mathbf{F}_q$ for some q. (For example, $p^{-1} + p^{-1/2} + p^{-1/3} + \cdots$ is an element of L which cannot be approached by elements of $\bar{\mathbf{Q}}_p$.) In fact, we can show that the set of series with these properties forms an algebraically closed field, using the following lemma, which is of interest in its own right, and which we can apply also toward the computation of the algebraic closure of Laurent series fields.

LEMMA 5. *Suppose E is an algebraically closed field, and $S \subseteq \mathrm{Aut}(E)$. Let F be the set of elements $e \in E$ whose orbit $\{\sigma(e) \mid \sigma \in S\}$ under S is finite. Then F is an algebraically closed subfield of E.*

Proof. Let $\mathrm{Orb}(x)$ denote the orbit of x under S. If $x, y \in F$, then $\mathrm{Orb}(x + y) \subseteq \mathrm{Orb}(x) + \mathrm{Orb}(y)$ which is finite, so $x + y \in F$. Similar considerations complete the proof that F is a subfield.

Given $p(x) \in F[x]$, let c be a zero of p in E. Then the orbit of $p(x)$ under S is finite (since each coefficient has finite orbit), and $\mathrm{Orb}(c)$ consists of zeros of polynomials in the orbit of $p(x)$ (to be specific, $\sigma(c)$ is a zero of $\sigma(p)$), so $c \in F$. Hence F is algebraically closed. □

The characteristic p case of the following corollary was proved by Rayner [16] using a different method.

COROLLARY 7. *If k is an algebraically closed field of characteristic 0, then $k((t)) = \cup_{n=1}^{\infty} k((t^{1/n}))$. If k is an algebraically closed field of characteristic p, then the set of series in $k((\mathbf{Q}))$ with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N (depending on the series) is an algebraically closed field containing $k((t))$.*

Proof. If ζ is a homomorphism from $\mathbf{Q}/\mathbf{Z}$ to the group of all roots of unity in k, then we get an automorphism of the algebraically closed Mal'cev-Neumann ring $k((\mathbf{Q}))$ by mapping $\sum_{q \in \mathbf{Q}} \alpha_q t^q$ to $\sum_{q \in \mathbf{Q}} \zeta(q)\alpha_q t^q$. Let $E = k((\mathbf{Q}))$ and let S be the set of all such automorphisms. Then the lemma shows that the set F of elements of E with finite orbit under S is an algebraically closed field. If $\operatorname{char} k = 0$, $F = \cup_{n=1}^{\infty} k((t^{1/n}))$, and the desired result follows easily. If $\operatorname{char} k = p$, F is the set of series in $k((\mathbf{Q}))$ with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N (since ζ is necessarily trivial on $\mathbf{Z}[1/p]/\mathbf{Z}$). □

COROLLARY 8. *The set of series in L with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N such that the residue classes of the coefficients lie in $\mathbf{F}_q$ for some q forms an algebraically closed field which contains $\mathbf{Q}_p$, hence also $\bar{\mathbf{Q}}_p$.*

Proof. If μ denotes the group of all k^{th} roots of 1 for all k relatively prime to p, and $\zeta: \mathbf{Q}/\mathbf{Z} \to \mu$ is any group homomorphism, then we get an automorphism of $A((\mathbf{Q}))$ (using the notation of Section 4) by sending $\sum_{g \in \mathbf{Q}} \alpha_g t^g$ to $\sum_{g \in \mathbf{Q}} \zeta(g)\alpha_g t^g$. This maps the ideal N into itself, so it induces an automorphism of L. We also get automorphisms of L coming functorially from the automorphisms of $\bar{\mathbf{F}}_p$.

Let $E = L$, and let S be the set of both types of automorphisms. Then the elements of L with finite orbit under the first type of automorphisms are those with support in $\frac{1}{N}\mathbf{Z}[1/p]$ for some N, and the elements with finite orbit under the second type of automorphisms are those such that the residue classes of the coefficients lie in $\mathbf{F}_q$ for some q. Hence the result follows from the lemma. (Obviously this field contains $\mathbf{Q}_p$.) □

There are many automorphisms of L besides those used in the previous proof. In fact, L has an enormous number of continuous automorphisms even over $\mathbf{C}_p$.

PROPOSITION 11. *Given* $\mu \in L/\mathbf{C}_p$, *let* $r = \sup_{e \in \mathbf{C}_p} v(\mu - e) \in \mathbf{R}$. *Then for any* $\mu' \in L$ *such that* $v(\mu - \mu') \geqslant r$, *there exists a continuous automorphism of* L *over* $\mathbf{C}_p$ *taking* μ *to* μ'.

Proof. We will extend the inclusion $\mathbf{C}_p \to L$ to an embedding $\mathbf{C}_p(\mu) \to L$ using the proof of Theorem 2 (instead of taking the obvious inclusion). There is no best approximation to μ in $\mathbf{C}_p$, since given any approximation, we can find a better one by subtracting the leading term of the series of the difference. So we are in Case 2 of the proof of Theorem 2, and it follows that we may embed $\mathbf{C}_p(\mu)$ in L by sending μ to any solution $\mu' \in L$ of the inequalities $v(x - e_\sigma) \geqslant g_\sigma$, where e_σ ranges over all elements of $\mathbf{C}_p$ and $g_\sigma = v(\mu - e_\sigma)$. These are satisfied if $v(\mu - \mu') \geqslant r$, by the triangle inequality. Finally, extend this embedding $\mathbf{C}_p(\mu) \to L$ to a continuous endomorphism $L \to L$ using Theorem 2. This endomorphism is an automorphism by Proposition 7. □

Lampert proved that $\mathbf{C}_p$ has transcendence degree $2^{\aleph_0}$ over the completion $\mathbf{C}_p^{\text{unram}}$ of the maximal unramified extension $\mathbf{Q}_p^{\text{unram}}$ of $\mathbf{Q}_p$, and that $\mathbf{C}_p^{\text{unram}}$ has transcendence degree $2^{\aleph_0}$ over $\mathbf{Q}_p$. We now extend this chain of results by calculating the transcendence degree of L over $\mathbf{C}_p$, using the following generalization of a proposition of Lampert's.

PROPOSITION 12. *If* V *is a sub-*$\mathbf{Q}$*-vector space of* $\mathbf{R}$ *containing* $\mathbf{Q}$, *then the set of elements in* L *of which all the accumulation values of the exponents are in* V *form a complete algebraically closed field.*

Proof. The proof is exactly the same as Lampert's proof for the special case $V = \mathbf{Q}$ [9]. □

COROLLARY 9. L *has transcendence degree* $2^{\aleph_0}$ *over* $\mathbf{C}_p$.

Proof. Let B be a basis for $\mathbf{R}$ as a vector space over $\mathbf{Q}$, with $1 \in B$. For each $b \in B, b \neq 1$, pick a strictly increasing sequence $q_1, q_2, \ldots$ in $\mathbf{Q}$ with limit b, and define $z_b = p^{q_1} + p^{q_2} + \cdots \in L$. Let K_b be the field of Proposition 12 with V the $\mathbf{Q}$-vector space generated by all elements of B except b. Then K_b contains $\mathbf{C}_p$, since it contains $\mathbf{Q}_p$ and is complete and algebraically closed. If $c \in B, z_c \in K_b$ iff $c \neq b$. But each K_b is algebraically closed, so no z_b can be algebraically dependent on the others over $\mathbf{C}_p$. Thus the transcendence degree of L over $\mathbf{C}_p$ exceeds the dimension of $\mathbf{R}$ over $\mathbf{Q}$ (it does not matter that we threw away one basis element), which is $2^{\aleph_0}$. On the other hand the cardinality of L is only $2^{\aleph_0}$, by Proposition 10. So the transcendence degree must equal $2^{\aleph_0}$. □

Traditionally, p-adic analysis has been done in $\mathbf{C}_p$. But every power series $F(X) = \sum_{n=0}^{\infty} a_n X^n$ with $a_n \in \mathbf{C}_p$ can be defined on L, and the radius of convergence is the same in L as in $\mathbf{C}_p$, because in either field the series converges iff the valuation of its terms approach $+\infty$. (Remember that L is complete.) As an example, we state the following proposition.

PROPOSITION 13. *There exists a unique function* $\log_p: L^* \to L$ *such that*

(1) $\log_p x = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n/n$, *for* $v(x-1) > 0$.

(2) $\log_p xy = \log_p x + \log_p y$, *for all* $x, y \in L^*$.

(3) $\log_p p = 0$.

Proof. The proof for L is exactly the same as the proof for $\mathbf{C}_p$. See pp. 87-88 in [7]. □

Although we can extend any power series defined on $\mathbf{C}_p$ to L, it seems that p-adic analysis rarely (if ever) would need to use properties of L not true of $\mathbf{C}_p$. All that seems important is that the field is a complete algebraically closed immediate extension of $\bar{\mathbf{Q}}_p$. It would be interesting to investigate whether anything can be gained by doing p-adic analysis in L instead of in $\mathbf{C}_p$.

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