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From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \dots, \varepsilon_{35} \in \{0, 1\}} \left[ \left( \sum_{j=1}^n \varepsilon_j z^j \right) + \left( \frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small  $\delta$ , we may apply Lemma 3.1 to deduce  $z \in \bar{W}$ .  $\square$

We now combine all the results of this section.

**THEOREM 3.1.** *There is an open neighborhood of  $\{z: |z| = 1, z \neq 1\}$  contained in  $\bar{W}$ .*

*Proof.* Apply Propositions 3.2 and 3.3.  $\square$

**COROLLARY 3.1.** *If  $z \in (-1, -1 + \delta)$  for sufficiently small  $\delta$  then  $z$  is a multiple zero of some 0, 1 power series.*

*Proof.* By Theorem 3.1, if  $\delta$  is small enough we can pick 0, 1 power series  $f_n$  and zeros  $z_n$  of  $f_n$  such that  $z_n \notin \mathbf{R}$  and  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . By taking a subsequence we may assume that the coefficient of  $z^k$  in  $f_n$  is eventually constant for large  $n$ , for each  $k$ . By a Rouché's Theorem argument, the pairs of zeros  $\{z_n, \bar{z}_n\}$  of  $f_n$  must converge to (at least) a double zero at  $z$  of  $\lim_{n \rightarrow \infty} f_n$ .  $\square$

#### 4. $\bar{W}$ IS CONNECTED

Since  $W$  is countable, we cannot hope to prove  $W$  is connected. We prove instead that  $\bar{W}$  is connected. First we need some topological lemmas.

Give  $\{0, 1\}$  the discrete topology and  $\{0, 1\}^\omega$  the product topology, as usual. If  $v = (v_1, v_2, \dots, v_n)$  is a finite vector of 0's and 1's, let  $S_v$  be the set of sequences in  $\{0, 1\}^\omega$  which start with  $v$ . The following lemma is the key ingredient in the connectivity proof.

**LEMMA 4.1.** *Let  $Y$  be a topological space. Suppose  $f: \{0, 1\}^\omega \rightarrow Y$  is a continuous map such that*

$$(4.1) \quad f(S_{v0}) \cap f(S_{v1}) \neq \emptyset$$

*for all  $v \in \{0, 1\}^n$ , and all  $n \geq 0$ . (Here  $v0$  denotes the vector  $v$  with 0 appended, etc.) Then the image of  $f$  is path connected.*

*Proof.* Let  $w(0) = f(x'_0)$  and  $w(1) = f(x_1)$  be elements of the image we wish to connect by a path. Find  $x_{1/2}, x'_{1/2} \in \{0, 1\}^\omega$  such that  $x'_0, x_{1/2}$  have the same first coordinate, and  $x'_{1/2}, x_1$  have the same first coordinate and  $f(x_{1/2}) = f(x'_{1/2})$ . (If  $x'_0, x_1$  have the same first coordinate, take  $x_{1/2} = x'_{1/2} = x'_0$ ; otherwise apply the hypothesis (4.1) with  $v$  as the empty vector.) Let  $w(1/2)$  be this common value.

Next find  $x_{1/4}, x'_{1/4} \in \{0, 1\}^\omega$ , using the same argument, such that  $x'_0, x_{1/4}$  agree in the first two coordinates,  $x'_{1/4}, x_{1/2}$  agree in the first two coordinates, and  $f(x_{1/4}) = f(x'_{1/4})$ . Let  $w(1/4)$  be this common value. Do the analogous thing at  $3/4$ .

By induction, we may continue to define  $x_{d/2^n}, x'_{d/2^n}, w(d/2^n)$  at all dyadic rationals  $d/2^n$  in  $[0, 1]$ , such that  $x'_{d/2^n}$  and  $x_{(d+1)/2^n}$  agree in the first  $n$  coordinates and

$$w(d/2^n) = f(x_{d/2^n}) = f(x'_{d/2^n}).$$

By induction, we see that all the  $x'_q$  with  $q \in [d/2^n, (d+1)/2^n)$  agree in the first  $n$  coordinates. Hence for

$$r = \sum_{i=1}^{\infty} \varepsilon_i 2^{-i} \in [0, 1], \quad \varepsilon_i \in \{0, 1\}$$

not a dyadic rational, we may define

$$x_r = x'_r = \lim_{n \rightarrow \infty} x'_{\sigma(n)} \quad \text{where} \quad \sigma(n) = \sum_{i=1}^n \varepsilon_i 2^{-i},$$

and  $w(r) = f(x_r)$ . Then  $w$  maps  $[d/2^n, (d+1)/2^n]$  into  $f(S_v)$  where  $v \in \{0, 1\}^n$  is the first  $n$  coordinates of  $x'_r$ ,  $r \in [d/2^n, (d+1)/2^n)$  and of  $x_{(d+1)/2^n}$ .

We now show that  $w$  is continuous at  $r \in [0, 1]$ . Let  $U$  be an open set of  $Y$  containing  $w(r)$ . Then  $f^{-1}(U)$  contains  $S_v$  and  $S_{v'}$  for some finite substrings  $v, v'$  of  $x_r, x'_r$  respectively, by continuity of  $f$ . By the last sentence of the previous paragraph it follows that

$$w^{-1}(U) \supseteq w^{-1}(f(S_v) \cup f(S_{v'}))$$

will contain a neighborhood of  $r$ .

Thus  $w: [0, 1] \rightarrow \text{image}(f)$  is a continuous path, and  $\text{image}(f)$  is path connected.  $\square$

Let  $M$  be a topological space. Give  $M^n$  the product topology and let the symmetric group  $S_n$  act on  $M^n$  by permuting the coordinates. The space

$M^n/S_n$ , which parameterizes  $n$ -element multisets, can be given the quotient topology.

LEMMA 4.2. *If  $A \subseteq M^n/S_n$  is connected, and the multiset  $\{P, P, \dots, P\}$  is in  $A$  for some  $P \in M$ , then the subset  $B \subseteq M$  of all coordinates of points in  $A$  is connected.*

*Proof.* Suppose not. Then there are open sets  $U, V \subseteq M$  such that  $U \cap B$  and  $V \cap B$  are disjoint nonempty sets with union  $B$ . Without loss of generality,  $P \in U$ . Let

$$\begin{aligned} U' &= U \times U \times \cdots \times U, \\ V' &= (V \times M \times M \times \cdots \times M) \\ &\quad \cup (M \times V \times M \times \cdots \times M) \\ &\quad \vdots \\ &\quad \cup (M \times M \times M \times \cdots \times V). \end{aligned}$$

Then  $U', V'$  are open sets in  $M^n$  which are stable under  $S_n$ , so they project to open sets  $U'', V''$  in  $M^n/S_n$ . Also  $A \subseteq U'' \cup V''$  since a point in  $A$  must have all coordinates in  $U$ , or else at least one coordinate in  $B \setminus U \subseteq V$ . Furthermore  $P \in U'' \cap A$ , and  $V'' \cap A$  is nonempty also, since at least one point of  $A$  has a coordinate in  $V$ , since  $V \cap B \neq \emptyset$ . Finally  $U'' \cap V'' \cap A = \emptyset$ , since it is not possible for a point of  $A$  to have all coordinates in  $U$ , yet have some coordinate in  $V$ . This contradicts the connectedness of  $A$ .  $\square$

THEOREM 4.1.  $\bar{W}$  is connected.

*Proof.* First we show that for  $\delta \in (0, 1)$ ,

$$\bar{W}_\delta = (\bar{W} \cap \{z: |z| \leq 1\}) \cup \{z: 1 - \delta \leq |z| \leq 1\}$$

is connected. The idea is to apply Lemma 4.1 to the function  $f$  which assigns to  $(\varepsilon_1, \varepsilon_2, \dots)$  the set of zeros of

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots$$

inside  $\{z: |z| < 1 - \delta\}$ . To make a continuous map of this requires some manipulation.

By Jensen's theorem, as was shown in Section 2, there is an upper bound  $n$  on the number of zeros that a power series with 0, 1 coefficients can have inside  $\{z: |z| < 1 - \delta\}$ . Let  $M$  be  $\{z: |z| \leq 1\}$  with the annulus

$\{z: 1 - \delta \leq |z| \leq 1\}$  shrunk to a point  $P$ . (Therefore  $M$  is topologically a sphere.) To each power series  $1 + \sum_{i=1}^{\infty} \varepsilon_i z^i$ ,  $\varepsilon_i \in \{0, 1\}$ , we assign the set of zeros inside  $\{z: |z| < 1 - \delta\}$ , (counted with multiplicities) and throw in extra copies of the point  $P$  as necessary to bring the total number of points to  $n$ . Since the order of these  $n$  elements of  $M$  is unspecified, we obtain a point of  $M^n/S_n$ . Let  $f((\varepsilon_1, \varepsilon_2, \dots))$  be this point.

We claim that this map

$$f: \{0, 1\}^{\omega} \rightarrow M^n/S_n$$

is continuous. This follows easily from Rouché's theorem; if two power series agree in the first  $m$  coordinates for  $m$  sufficiently large then their zeros inside  $\{z: |z| < 1 - \delta\}$  will be within  $\varepsilon$ . Some may escape or enter the disk, but this is not a problem, since in the topology on  $M$ ,  $P$  is close to all points  $z$  with  $|z|$  sufficiently near  $1 - \delta$ .

We next check condition (4.1) of Lemma 4.1. This is easily done using the following trick: given

$$v = (v_1, v_2, \dots, v_n) \in \{0, 1\}^n,$$

let  $w = (v_1, v_2, \dots, v_n, 1, v_1, v_2, \dots, v_n)$ . Then  $v \in S_{v_0}$ ,  $w \in S_{v_1}$ , and  $f(v) = f(w)$  (we extend  $v, w$  to infinite vectors by appending 0's), since

$$1 + v_1 z + v_2 z^2 + \dots + v_n z^n$$

and

$$\begin{aligned} &1 + v_1 z + v_2 z^2 + \dots + v_n z^n + z^{n+1} + v_1 z^{n+2} + \dots + v_n z^{2n+1} \\ &= (1 + v_1 z + v_2 z^2 + \dots + v_n z^n) (1 + z^{n+1}) \end{aligned}$$

have the same zeros inside  $\{z: |z| < 1 - \delta\}$ . Therefore we may apply Lemma 4.1 and deduce that the image of  $f$  is path connected.

Since  $f((0, 0, \dots)) = (P, P, P, \dots, P)$ , we may apply Lemma 4.2 with  $A = \text{image}(f)$  to deduce that  $\bar{W}_\delta$  with the annulus  $\{z: 1 - \delta \leq |z| \leq 1\}$  shrunk to a point  $P$  is a connected subset of  $M$ . This is equivalent to the connectivity of  $\bar{W}_\delta$ .

Since  $\bar{W} \cap \{z: |z| \leq 1\}$  is the decreasing intersection of the compact connected sets  $\bar{W}_{1/m}$ , it too is connected. So is its image under  $z \mapsto 1/z$ . Finally,  $\bar{W}$  is the union of these two sets, which meet on the unit circle, so  $\bar{W}$  is connected as well.  $\square$