

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ZEROS OF POLYNOMIALS WITH 0, 1 COEFFICIENTS
Autor: Odlyzko, A. M. / Poonen, B.
Kapitel: 3. A NEIGHBORHOOD OF THE UNIT CIRCLE
DOI: <https://doi.org/10.5169/seals-60430>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

To show that $f_{m,n}(z)$ has a zero β near $\alpha = \alpha_{m,n}$, let

$$(2.18) \quad g(z) = m + z^n.$$

Then $g(\alpha) = 0$. Consider the circle $|z - \alpha| = (10n)^{-1}$. On this circle, $|g(z)| \geq m/100$, while

$$(2.19) \quad |(1 + z + \cdots + z^{m-1}) - m| \leq \sum_{k=1}^{m-1} |z^k - 1| = O(m^2/n),$$

so for $m = o(n)$, by Rouché's theorem $g(z)$ and $f_{m,n}(z)$ have the same number of zeros inside the circle, namely one. This proves the claim and answers the Conway-Parker question. \square

3. A NEIGHBORHOOD OF THE UNIT CIRCLE

In this section we prove that an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ is contained in \bar{W} .

LEMMA 3.1. *If $B \subseteq \mathbf{C}$ is compact, $n \geq 1, |z| < 1$, and*

$$(3.1) \quad B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{i=1}^n \varepsilon_i z^i \right) + z^n B \right],$$

then every element of B is expressible in the form

$$(3.2) \quad \sum_{i=1}^{\infty} \varepsilon_i z^i, \quad \varepsilon_i \in \{0, 1\}.$$

In particular, if $-1 \in B$, then $z \in \bar{W}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $\varepsilon_{mi} \in \{0, 1\}$, $m \geq 0, 1 \leq i \leq n$ such that

$$b_m = \left(\sum_{i=1}^n \varepsilon_{mi} z^i \right) + z^n b_{m+1}.$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^n \varepsilon_{mi} z^{mn+i} \right) + z^{Mn} b_M.$$

Since B is compact, $z^{Mn} b_M \rightarrow 0$ as $M \rightarrow \infty$, so

$$b_0 = \sum_{m=0}^{\infty} \sum_{i=1}^n \varepsilon_{mi} z^{mn+i},$$

which is the desired form. \square

PROPOSITION 3.1. *If $z \in \mathbf{R}$, $-1 < z \leq -\varphi^{-1}$, then $z \in \bar{W}$.*

Proof. Let $B = [-1, -z]$. Then, since $-1 < z \leq -\varphi^{-1}$ implies $z - z^2 \leq -1$, we have

$$\begin{aligned} (z + zB) \cup zB &= [z - z^2, 0] \cup [-z^2, -z] \\ &= [z - z^2, -z] \\ &\supseteq [-1, -z] \\ &= B. \end{aligned}$$

We now apply Lemma 3.1 with $n = 1$, and conclude that $z \in \bar{W}$. \square

LEMMA 3.2. *If $B \subseteq \mathbf{C}$ is compact, $-1 \in B$, $n \geq 1$, $x \in \mathbf{C}$ and*

$$(3.3) \quad B \subseteq \text{int} \bigcup_{\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{i=1}^n \varepsilon_i x^i \right) + x^n B \right],$$

where $\text{int } S$ denotes the interior of S , then there is a neighborhood N of x such that

$$N \cap \{z : |z| < 1\} \subseteq \bar{W}.$$

Proof. Condition (3.3) implies that (3.1) holds for z in a neighborhood of x , so Lemma 3.2 follows from Lemma 3.1. \square

LEMMA 3.3. *If $B = \{z : |z| \leq R\}$ for some $R \geq 1$, $n \geq 1$, $|x| = 1$ and*

$$(3.4) \quad B \subseteq \text{int} \bigcup_{j=1}^n (x^j + B),$$

then

$$x \in \text{int } \bar{W}.$$

Proof. Since \bar{W} contains the unit circle and is closed under $z \mapsto 1/z$, this follows trivially from Lemma 3.2. \square

PROPOSITION 3.2. *If $|x| = 1$, $x \neq \pm 1$, then $x \in \text{int } \bar{W}$.*

Proof. We claim that if $R \geq 2$, then the condition (3.4) of Lemma 3.3 holds for n large enough. If $x = \exp(\pi i\alpha)$ and α is irrational, then by Kronecker's theorem $\{x^j : j \geq 1\}$ is dense on the unit circle, and then for every $\delta > 0$, the disk of radius $R + 1 - \delta$ is contained in the union on the right side of (3.4) for n large enough. If α is rational, then the $\{x^j : j \geq 1\}$ are the vertices of a regular k -gon, and $k \geq 3$ since $x \neq \pm 1$. In that case the union on the right side of (3.4) contains a disk of radius r , where r , 1, and R are the sides of a triangle, and the angle between the sides of lengths r and 1 is π/k . Therefore, by the Law of Cosines,

$$R^2 = 1 + r^2 - 2r \cos(\pi/k),$$

and so

$$r = \cos(\pi/k) + (\cos^2(\pi/k) + R^2 - 1)^{1/2}.$$

Since $\cos(\pi/k) \geq \cos(\pi/3)$, we find that

$$r \geq 1/4 + (R^2 - 3/4)^{1/2} \geq R + 1/20$$

for $R \geq 2$, since $(R^2 - 3/4)^{1/2} - R$ is an increasing function of R . \square

Proving $-1 \in \text{int } \bar{W}$ is trickier, because it will not do to take B as a disc of radius ≥ 1 if $\text{Im}(z)$ is small compared to $\text{Re}(z+1)$. We will instead take B as a parallelogram that becomes flatter and flatter as $\text{Im } z \rightarrow 0$. The following two lemmas will be used in verifying the condition of Lemma 3.1.

LEMMA 3.4. Let $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. Let $v_j = T^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)^j \begin{pmatrix} 1 \\ -j \end{pmatrix}$. Then for $n \geq 16$

$$\left\{ \sum_{j=1}^n \varepsilon_j v_j : \varepsilon_j \in \{0, 1\} \right\}$$

contains $\{(a, b) : a, b \in \mathbf{Z}, |a| \leq 1, |b| \leq n - 16\}$.

Proof. Given such $\begin{pmatrix} a \\ b \end{pmatrix}$, first pick $\varepsilon_1, \varepsilon_2$ so that $\varepsilon_1 v_1 + \varepsilon_2 v_2$ has first coordinate a . Next pick $\varepsilon_3 = \varepsilon_4 = 0$ or 1 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4$ has first coordinate a and second coordinate b' with $b' \not\equiv b \pmod{2}$. Certainly $|b'| \leq 1 + 2 + 3 + 4 = 10$, so $|b - b'| \leq n - 6$. If $b > b'$, then

$$\begin{aligned} \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_5 + v_{5+b-b'} \\ = (a, b') - (1, -5) + (1, 5 + b - b') \\ = (a, b). \end{aligned}$$

If $b < b'$, then

$$\begin{aligned} \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_6 + v_{6+b'-b} \\ = (a, b') + (1, -6) - (1, 6 + b' - b) \\ = (a, b). \quad \square \end{aligned}$$

LEMMA 3.5. Let T, v_j be as in Lemma 3.4. Let B be the square with vertices $(\pm 1, \pm 1)$. Then for $n \geq 35$,

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{j=1}^n \varepsilon_j v_j \right) + \frac{1}{2} T^n B \right].$$

Proof. $\frac{1}{2} T^n$ is the parallelogram with vertices

$$\pm \frac{1}{2} (1, -n) \pm \frac{1}{2} (0, 1).$$

The cross-section of this with x -coordinate x_0 is the vertical interval $[-nx_0 - 1/2, -nx_0 + 1/2]$ for $-1/2 \leq x_0 \leq 1/2$. Hence given $(\alpha, \beta) \in B$ pick $a \in \{-1, 0, 1\}$ such that $-1/2 \leq \alpha + a \leq 1/2$ and then pick $b \in \mathbf{Z}$ such that $-n(\alpha + a) + 1/2 \leq \beta + b \leq -n(\alpha + a) + 1/2$. Since $|\beta| \leq 1$ and $|\alpha + a| \leq 1/2$, we see $|b| \leq \frac{1}{2}(n+1) + 1 \leq n - 16$ if $n \geq 35$. Then $(\alpha, \beta) + (\alpha, b) \in \frac{1}{2} T^n B$ and by Lemma 3.4 we can pick $\varepsilon_1, \dots, \varepsilon_n$ such that $\sum_{j=1}^n \varepsilon_j v_j = -(\alpha, b)$, so Lemma 3.5 follows. \square

PROPOSITION 3.3. $-1 \in \text{int } \bar{W}$.

Proof. Since \bar{W} is closed under $z \mapsto \bar{z}$ and $z \mapsto 1/z$, it suffices to show that for $|z| < 1$, $\text{Im } z > 0$, and $|z + 1|$ sufficiently small, z is in \bar{W} . (Proposition 3.1 handles the case $z \in \mathbf{R}$.) Let $\delta = z + 1$. Let B be the parallelogram with vertices $\pm 1 \pm \delta$.

We work in a nonstandard coordinate system for \mathbf{C} , with basis vectors 1 and δ , so B is represented by the square with vertices $(\pm 1, \pm 1)$. We claim that multiplication by z is represented by the matrix $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ up to $O(|\delta|)$. We have

$$\begin{aligned} z \cdot 1 &= -1 + \delta \\ z \cdot \delta &= -\delta + \delta^2 \end{aligned}$$

and

$$\delta^2 - 2(\text{Re } \delta)\delta + |\delta|^2 = 0$$

so δ^2 corresponds to $(|\delta|^2, -2 \text{Re } \delta)$ in our basis, and is $O(|\delta|)$.

From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \dots, \varepsilon_{35} \in \{0, 1\}} \left[\left(\sum_{j=1}^n \varepsilon_j z^j \right) + \left(\frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small δ , we may apply Lemma 3.1 to deduce $z \in \bar{W}$. \square

We now combine all the results of this section.

THEOREM 3.1. *There is an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ contained in \bar{W} .*

Proof. Apply Propositions 3.2 and 3.3. \square

COROLLARY 3.1. *If $z \in (-1, -1 + \delta)$ for sufficiently small δ then z is a multiple zero of some 0, 1 power series.*

Proof. By Theorem 3.1, if δ is small enough we can pick 0, 1 power series f_n and zeros z_n of f_n such that $z_n \notin \mathbf{R}$ and $z_n \rightarrow z$ as $n \rightarrow \infty$. By taking a subsequence we may assume that the coefficient of z^k in f_n is eventually constant for large n , for each k . By a Rouché's Theorem argument, the pairs of zeros $\{z_n, \bar{z}_n\}$ of f_n must converge to (at least) a double zero at z of $\lim_{n \rightarrow \infty} f_n$. \square

4. \bar{W} IS CONNECTED

Since W is countable, we cannot hope to prove W is connected. We prove instead that \bar{W} is connected. First we need some topological lemmas.

Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology, as usual. If $v = (v_1, v_2, \dots, v_n)$ is a finite vector of 0's and 1's, let S_v be the set of sequences in $\{0, 1\}^\omega$ which start with v . The following lemma is the key ingredient in the connectivity proof.

LEMMA 4.1. *Let Y be a topological space. Suppose $f: \{0, 1\}^\omega \rightarrow Y$ is a continuous map such that*

$$(4.1) \quad f(S_{v_0}) \cap f(S_{v_1}) \neq \emptyset$$

for all $v \in \{0, 1\}^n$, and all $n \geq 0$. (Here v_0 denotes the vector v with 0 appended, etc.) Then the image of f is path connected.