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To show that $f_{m,n}(z)$ has a zero β near $\alpha = \alpha_{m,n}$, let

$$(2.18) \quad g(z) = m + z^n .$$

Then $g(\alpha) = 0$. Consider the circle $|z - \alpha| = (10n)^{-1}$. On this circle, $|g(z)| \geq m/100$, while

$$(2.19) \quad |(1 + z + \cdots + z^{m-1}) - m| \leq \sum_{k=1}^{m-1} |z^k - 1| = O(m^2/n) ,$$

so for $m = o(n)$, by Rouché's theorem $g(z)$ and $f_{m,n}(z)$ have the same number of zeros inside the circle, namely one. This proves the claim and answers the Conway-Parker question. \square

3. A NEIGHBORHOOD OF THE UNIT CIRCLE

In this section we prove that an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ is contained in \bar{W} .

LEMMA 3.1. *If $B \subseteq \mathbb{C}$ is compact, $n \geq 1, |z| < 1$, and*

$$(3.1) \quad B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{i=1}^n \varepsilon_i z^i \right) + z^n B \right] ,$$

then every element of B is expressible in the form

$$(3.2) \quad \sum_{i=1}^{\infty} \varepsilon_i z^i, \quad \varepsilon_i \in \{0, 1\} .$$

In particular, if $-1 \in B$, then $z \in \bar{W}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $\varepsilon_{mi} \in \{0, 1\}$, $m \geq 0, 1 \leq i \leq n$ such that

$$b_m = \left(\sum_{i=1}^n \varepsilon_{mi} z^i \right) + z^n b_{m+1} .$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^n \varepsilon_{mi} z^{mn+i} \right) + z^{Mn} b_M .$$

Since B is compact, $z^{Mn} b_M \rightarrow 0$ as $M \rightarrow \infty$, so

$$b_0 = \sum_{m=0}^{\infty} \sum_{i=1}^n \varepsilon_{mi} z^{mn+i},$$

which is the desired form. \square

PROPOSITION 3.1. *If $z \in \mathbf{R}$, $-1 < z \leq -\varphi^{-1}$, then $z \in \bar{W}$.*

Proof. Let $B = [-1, -z]$. Then, since $-1 < z \leq -\varphi^{-1}$ implies $z - z^2 \leq -1$, we have

$$\begin{aligned} (z + zB) \cup zB &= [z - z^2, 0] \cup [-z^2, -z] \\ &= [z - z^2, -z] \\ &\supseteq [-1, -z] \\ &= B. \end{aligned}$$

We now apply Lemma 3.1 with $n = 1$, and conclude that $z \in \bar{W}$. \square

LEMMA 3.2. *If $B \subseteq \mathbf{C}$ is compact, $-1 \in B$, $n \geq 1$, $x \in \mathbf{C}$ and*

$$(3.3) \quad B \subseteq \text{int} \bigcup_{\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{i=1}^n \varepsilon_i x^i \right) + x^n B \right],$$

where $\text{int } S$ denotes the interior of S , then there is a neighborhood N of x such that

$$N \cap \{z : |z| < 1\} \subseteq \bar{W}.$$

Proof. Condition (3.3) implies that (3.1) holds for z in a neighborhood of x , so Lemma 3.2 follows from Lemma 3.1. \square

LEMMA 3.3. *If $B = \{z : |z| \leq R\}$ for some $R \geq 1$, $n \geq 1$, $|x| = 1$ and*

$$(3.4) \quad B \subseteq \text{int} \bigcup_{j=1}^n (x^j + B),$$

then

$$x \in \text{int } \bar{W}.$$

Proof. Since \bar{W} contains the unit circle and is closed under $z \mapsto 1/z$, this follows trivially from Lemma 3.2. \square

PROPOSITION 3.2. *If $|x| = 1$, $x \neq \pm 1$, then $x \in \text{int } \bar{W}$.*

Proof. We claim that if $R \geq 2$, then the condition (3.4) of Lemma 3.3 holds for n large enough. If $x = \exp(\pi i \alpha)$ and α is irrational, then by Kronecker's theorem $\{x^j : j \geq 1\}$ is dense on the unit circle, and then for every $\delta > 0$, the disk of radius $R + 1 - \delta$ is contained in the union on the right side of (3.4) for n large enough. If α is rational, then the $\{x^j : j \geq 1\}$ are the vertices of a regular k -gon, and $k \geq 3$ since $x \neq \pm 1$. In that case the union on the right side of (3.4) contains a disk of radius r , where r , 1, and R are the sides of a triangle, and the angle between the sides of lengths r and 1 is π/k . Therefore, by the Law of Cosines,

$$R^2 = 1 + r^2 - 2r \cos(\pi/k),$$

and so

$$r = \cos(\pi/k) + (\cos^2(\pi/k) + R^2 - 1)^{1/2}.$$

Since $\cos(\pi/k) \geq \cos(\pi/3)$, we find that

$$r \geq 1/4 + (R^2 - 3/4)^{1/2} \geq R + 1/20$$

for $R \geq 2$, since $(R^2 - 3/4)^{1/2} - R$ is an increasing function of R . □

Proving $-1 \in \text{int } \bar{W}$ is trickier, because it will not do to take B as a disc of radius ≥ 1 if $\text{Im}(z)$ is small compared to $\text{Re}(z + 1)$. We will instead take B as a parallelogram that becomes flatter and flatter as $\text{Im } z \rightarrow 0$. The following two lemmas will be used in verifying the condition of Lemma 3.1.

LEMMA 3.4. Let $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. Let $v_j = T^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)^j \begin{pmatrix} 1 \\ -j \end{pmatrix}$. Then for $n \geq 16$

$$\left\{ \sum_{j=1}^n \varepsilon_j v_j : \varepsilon_j \in \{0, 1\} \right\}$$

contains $\{ \binom{a}{b} : a, b \in \mathbf{Z}, |a| \leq 1, |b| \leq n - 16 \}$.

Proof. Given such $\binom{a}{b}$, first pick $\varepsilon_1, \varepsilon_2$ so that $\varepsilon_1 v_1 + \varepsilon_2 v_2$ has first coordinate a . Next pick $\varepsilon_3 = \varepsilon_4 = 0$ or 1 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4$ has first coordinate a and second coordinate b' with $b' \not\equiv b \pmod{2}$. Certainly $|b'| \leq 1 + 2 + 3 + 4 = 10$, so $|b - b'| \leq n - 6$. If $b > b'$, then

$$\begin{aligned} & \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_5 + v_{5+b-b'} \\ &= (a, b') - (1, -5) + (1, 5 + b - b') \\ &= (a, b). \end{aligned}$$

If $b < b'$, then

$$\begin{aligned} \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_6 + v_{6+b'-b} \\ = (a, b') + (1, -6) - (1, 6 + b' - b) \\ = (a, b). \quad \square \end{aligned}$$

LEMMA 3.5. Let T, v_j be as in Lemma 3.4. Let B be the square with vertices $(\pm 1, \pm 1)$. Then for $n \geq 35$,

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{j=1}^n \varepsilon_j v_j \right) + \frac{1}{2} T^n B \right].$$

Proof. $\frac{1}{2} T^n$ is the parallelogram with vertices

$$\pm \frac{1}{2} (1, -n) \pm \frac{1}{2} (0, 1).$$

The cross-section of this with x -coordinate x_0 is the vertical interval $[-nx_0 - 1/2, -nx_0 + 1/2]$ for $-1/2 \leq x_0 \leq 1/2$. Hence given $(\alpha, \beta) \in B$ pick $a \in \{-1, 0, 1\}$ such that $-1/2 \leq \alpha + a \leq 1/2$ and then pick $b \in \mathbf{Z}$ such that $-n(\alpha + a) + 1/2 \leq \beta + b \leq -n(\alpha + a) + 1/2$. Since $|\beta| \leq 1$ and $|\alpha + a| \leq 1/2$, we see $|b| \leq \frac{1}{2}(n+1) + 1 \leq n - 16$ if $n \geq 35$. Then $(\alpha, \beta) + (a, b) \in \frac{1}{2} T^n B$ and by Lemma 3.4 we can pick $\varepsilon_1, \dots, \varepsilon_n$ such that $\sum_{j=1}^n \varepsilon_j v_j = -(a, b)$, so Lemma 3.5 follows. \square

PROPOSITION 3.3. $-1 \in \text{int } \bar{W}$.

Proof. Since \bar{W} is closed under $z \mapsto \bar{z}$ and $z \mapsto 1/z$, it suffices to show that for $|z| < 1$, $\text{Im } z > 0$, and $|z + 1|$ sufficiently small, z is in \bar{W} . (Proposition 3.1 handles the case $z \in \mathbf{R}$.) Let $\delta = z + 1$. Let B be the parallelogram with vertices $\pm 1 \pm \delta$.

We work in a nonstandard coordinate system for \mathbf{C} , with basis vectors 1 and δ , so B is represented by the square with vertices $(\pm 1, \pm 1)$. We claim that multiplication by z is represented by the matrix $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ up to $O(|\delta|)$. We have

$$\begin{aligned} z \cdot 1 &= -1 + \delta \\ z \cdot \delta &= -\delta + \delta^2 \end{aligned}$$

and

$$\delta^2 - 2(\text{Re } \delta)\delta + |\delta|^2 = 0$$

so δ^2 corresponds to $(|\delta|^2, -2 \text{Re } \delta)$ in our basis, and is $O(|\delta|)$.

From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \dots, \varepsilon_{35} \in \{0, 1\}} \left[\left(\sum_{j=1}^n \varepsilon_j z^j \right) + \left(\frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small δ , we may apply Lemma 3.1 to deduce $z \in \bar{W}$. \square

We now combine all the results of this section.

THEOREM 3.1. *There is an open neighborhood of $\{z: |z|=1, z \neq 1\}$ contained in \bar{W} .*

Proof. Apply Propositions 3.2 and 3.3. \square

COROLLARY 3.1. *If $z \in (-1, -1 + \delta)$ for sufficiently small δ then z is a multiple zero of some 0, 1 power series.*

Proof. By Theorem 3.1, if δ is small enough we can pick 0, 1 power series f_n and zeros z_n of f_n such that $z_n \notin \mathbf{R}$ and $z_n \rightarrow z$ as $n \rightarrow \infty$. By taking a subsequence we may assume that the coefficient of z^k in f_n is eventually constant for large n , for each k . By a Rouché's Theorem argument, the pairs of zeros $\{z_n, \bar{z}_n\}$ of f_n must converge to (at least) a double zero at z of $\lim_{n \rightarrow \infty} f_n$. \square

4. \bar{W} IS CONNECTED

Since W is countable, we cannot hope to prove W is connected. We prove instead that \bar{W} is connected. First we need some topological lemmas.

Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology, as usual. If $v = (v_1, v_2, \dots, v_n)$ is a finite vector of 0's and 1's, let S_v be the set of sequences in $\{0, 1\}^\omega$ which start with v . The following lemma is the key ingredient in the connectivity proof.

LEMMA 4.1. *Let Y be a topological space. Suppose $f: \{0, 1\}^\omega \rightarrow Y$ is a continuous map such that*

$$(4.1) \quad f(S_{v0}) \cap f(S_{v1}) \neq \emptyset$$

for all $v \in \{0, 1\}^n$, and all $n \geq 0$. (Here $v0$ denotes the vector v with 0 appended, etc.) Then the image of f is path connected.