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To show that $f_{m,n}(z)$ has a zero β near $\alpha = \alpha_{m,n}$, let

$$g(z) = m + z^n . \tag{2.18}$$

Then $g(\alpha) = 0$. Consider the circle $|z - \alpha| = (10n)^{-1}$. On this circle, $|g(z)| \geqslant m/100$, while

$$|(1 + z + \cdots + z^{m-1}) - m| \leqslant \sum_{k=1}^{m-1} |z^k - 1| = O(m^2/n) , \tag{2.19}$$

so for $m = o(n)$, by Rouché's theorem $g(z)$ and $f_{m,n}(z)$ have the same number of zeros inside the circle, namely one. This proves the claim and answers the Conway-Parker question. □

3. A NEIGHBORHOOD OF THE UNIT CIRCLE

In this section we prove that an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ is contained in $\bar{W}$.

LEMMA 3.1. *If* $B \subseteq \mathbf{C}$ *is compact,* $n \geqslant 1, |z| < 1$, *and*

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0,1\}} \left[\left(\sum_{i=1}^{n} \varepsilon_i z^i \right) + z^n B \right] , \tag{3.1}$$

then every element of B *is expressible in the form*

$$\sum_{i=1}^{\infty} \varepsilon_i z^i, \qquad \varepsilon_i \in \{0, 1\} . \tag{3.2}$$

In particular, if $-1 \in B$, *then* $z \in \bar{W}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $\varepsilon_{mi} \in \{0, 1\}$, $m \geqslant 0$, $1 \leqslant i \leqslant n$ such that

$$b_m = \left(\sum_{i=1}^{n} \varepsilon_{mi} z^i \right) + z^n b_{m+1} .$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^{n} \varepsilon_{mi} z^{mn+i} \right) + z^{Mn} b_M .$$

Since B is compact, $z^{Mn} b_M \to 0$ as $M \to \infty$, so

$$b_0 = \sum_{m=0}^{\infty} \sum_{i=1}^{n} \varepsilon_{mi} z^{mn+i},$$

which is the desired form. □

PROPOSITION 3.1. *If* $z \in \mathbf{R}$, $-1 < z \leqslant -\varphi^{-1}$, *then* $z \in \bar{W}$.

Proof. Let $B = [-1, -z]$. Then, since $-1 < z \leqslant -\varphi^{-1}$ implies $z - z^2 \leqslant -1$, we have

$$\begin{aligned}(z + zB) \cup zB &= [z - z^2, 0] \cup [-z^2, -z] \\ &= [z - z^2, -z] \\ &\supseteq [-1, -z] \\ &= B .\end{aligned}$$

We now apply Lemma 3.1 with $n = 1$, and conclude that $z \in \bar{W}$. □

LEMMA 3.2. *If* $B \subseteq \mathbf{C}$ *is compact,* $-1 \in B$, $n \geqslant 1$, $x \in \mathbf{C}$ *and*

$$B \subseteq \operatorname{int} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{i=1}^{n} \varepsilon_i x^i\right) + x^n B\right], \tag{3.3}$$

where $\operatorname{int} S$ *denotes the interior of* S, *then there is a neighborhood* N *of* x *such that*

$$N \cap \{z : |z| < 1\} \subseteq \bar{W} .$$

Proof. Condition (3.3) implies that (3.1) holds for z in a neighborhood of x, so Lemma 3.2 follows from Lemma 3.1. □

LEMMA 3.3. *If* $B = \{z : |z| \leqslant R\}$ *for some* $R \geqslant 1$, $n \geqslant 1$, $|x| = 1$ *and*

$$B \subseteq \operatorname{int} \bigcup_{j=1}^{n} (x^j + B) , \tag{3.4}$$

then

$$x \in \operatorname{int} \bar{W} .$$

Proof. Since $\bar{W}$ contains the unit circle and is closed under $z \mapsto 1/z$, this follows trivially from Lemma 3.2. □

PROPOSITION 3.2. *If* $|x| = 1$, $x \neq \pm 1$, *then* $x \in \operatorname{int} \bar{W}$.

Proof. We claim that if $R \geqslant 2$, then the condition (3.4) of Lemma 3.3 holds for n large enough. If $x = \exp(\pi i \alpha)$ and α is irrational, then by Kronecker's theorem $\{x^j : j \geqslant 1\}$ is dense on the unit circle, and then for every $\delta > 0$, the disk of radius $R + 1 - \delta$ is contained in the union on the right side of (3.4) for n large enough. If α is rational, then the $\{x^j : j \geqslant 1\}$ are the vertices of a regular k-gon, and $k \geqslant 3$ since $x \neq \pm 1$. In that case the union on the right side of (3.4) contains a disk of radius r, where r, 1, and R are the sides of a triangle, and the angle between the sides of lengths r and 1 is π/k. Therefore, by the Law of Cosines,

$$R^2 = 1 + r^2 - 2r\cos(\pi/k)\,,$$

and so

$$r = \cos(\pi/k) + (\cos^2(\pi/k) + R^2 - 1)^{1/2}\,.$$

Since $\cos(\pi/k) \geqslant \cos(\pi/3)$, we find that

$$r \geqslant 1/4 + (R^2 - 3/4)^{1/2} \geqslant R + 1/20$$

for $R \geqslant 2$, since $(R^2 - 3/4)^{1/2} - R$ is an increasing function of R. □

Proving $-1 \in \operatorname{int} \overline{W}$ is trickier, because it will not do to take B as a disc of radius $\geqslant 1$ if $\operatorname{Im}(z)$ is small compared to $\operatorname{Re}(z + 1)$. We will instead take B as a parallelogram that becomes flatter and flatter as $\operatorname{Im} z \to 0$. The following two lemmas will be used in verifying the condition of Lemma 3.1.

LEMMA 3.4. *Let* $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. *Let* $v_j = T^j\binom{1}{0} = (-1)^j\binom{1}{-j}$. *Then for* $n \geqslant 16$

$$\left\{ \sum_{j=1}^{n} \varepsilon_j v_j : \varepsilon_j \in \{0, 1\} \right\}$$

contains $\{\binom{a}{b} : a, b \in \mathbf{Z}, |a| \leqslant 1, |b| \leqslant n - 16\}$.

Proof. Given such $\binom{a}{b}$, first pick ε_1, ε_2 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2$ has first coordinate a. Next pick $\varepsilon_3 = \varepsilon_4 = 0$ or 1 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4$ has first coordinate a and second coordinate b' with $b' \not\equiv b \bmod 2$. Certainly $|b'| \leqslant 1 + 2 + 3 + 4 = 10$, so $|b - b'| \leqslant n - 6$. If $b > b'$, then

$$\begin{aligned} &\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_5 + v_{5+b-b'} \\ &\quad = (a, b') - (1, -5) + (1, 5 + b - b') \\ &\quad = (a, b)\,. \end{aligned}$$

If $b < b'$, then

$$\begin{aligned} &\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_6 + v_{6+b'-b} \\ &\quad = (a, b') + (1, -6) - (1, 6 + b' - b) \\ &\quad = (a, b) . \quad \square \end{aligned}$$

LEMMA 3.5. *Let* T, v_j *be as in Lemma 3.4. Let* B *be the square with vertices* $(\pm 1, \pm 1)$. *Then for* $n \geqslant 35$,

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0,1\}} \left[\left(\sum_{j=1}^{n} \varepsilon_j v_j \right) + \frac{1}{2} T^n B \right] .$$

Proof. $\frac{1}{2} T^n$ is the parallelogram with vertices

$$\pm \frac{1}{2} (1, -n) \pm \frac{1}{2} (0, 1) .$$

The cross-section of this with x-coordinate x_0 is the vertical interval $[-nx_0 - 1/2, -nx_0 + 1/2]$ for $-1/2 \leqslant x_0 \leqslant 1/2$. Hence given $(\alpha, \beta) \in B$ pick $a \in \{-1, 0, 1\}$ such that $-1/2 \leqslant \alpha + a \leqslant 1/2$ and then pick $b \in \mathbf{Z}$ such that $-n(\alpha + a) + 1/2 \leqslant \beta + b \leqslant -n(\alpha + a) + 1/2$. Since $|\beta| \leqslant 1$ and $|\alpha + a| \leqslant 1/2$, we see $|b| \leqslant \frac{1}{2}(n+1) + 1 \leqslant n - 16$ if $n \geqslant 35$. Then $(\alpha, \beta) + (a, b) \in \frac{1}{2} T^n B$ and by Lemma 3.4 we can pick $\varepsilon_1, \ldots, \varepsilon_n$ such that $\sum_{j=1}^{n} \varepsilon_j v_j = -(a, b)$, so Lemma 3.5 follows. $\square$

PROPOSITION 3.3. $-1 \in \operatorname{int} \overline{W}$.

Proof. Since $\overline{W}$ is closed under $z \mapsto \bar{z}$ and $z \mapsto 1/z$, it suffices to show that for $|z| < 1$, $\operatorname{Im} z > 0$, and $|z + 1|$ sufficiently small, z is in $\overline{W}$. (Proposition 3.1 handles the case $z \in \mathbf{R}$.) Let $\delta = z + 1$. Let B be the parallelogram with vertices $\pm 1 \pm \delta$.

We work in a nonstandard coordinate system for $\mathbf{C}$, with basis vectors 1 and δ, so B is represented by the square with vertices $(\pm 1, \pm 1)$. We claim that multiplication by z is represented by the matrix $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ up to $O(|\delta|)$. We have

$$\begin{aligned} z \cdot 1 &= -1 + \delta \\ z \cdot \delta &= -\delta + \delta^2 \end{aligned}$$

and

$$\delta^2 - 2(\operatorname{Re} \delta)\delta + |\delta|^2 = 0$$

so δ^2 corresponds to $(|\delta|^2, -2 \operatorname{Re} \delta)$ in our basis, and is $O(|\delta|)$.

From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \ldots, \varepsilon_{35} \in \{0,1\}} \left[\left(\sum_{j=1}^{n} \varepsilon_j z^j \right) + \left(\frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small δ, we may apply Lemma 3.1 to deduce $z \in \bar{W}$. □

We now combine all the results of this section.

THEOREM 3.1. *There is an open neighborhood of* $\{z : |z| = 1, z \neq 1\}$ *contained in* $\bar{W}$.

Proof. Apply Propositions 3.2 and 3.3. □

COROLLARY 3.1. *If* $z \in (-1, -1+\delta)$ *for sufficiently small* δ *then* z *is a multiple zero of some* 0, 1 *power series.*

Proof. By Theorem 3.1, if δ is small enough we can pick 0, 1 power series f_n and zeros z_n of f_n such that $z_n \notin \mathbf{R}$ and $z_n \to z$ as $n \to \infty$. By taking a subsequence we may assume that the coefficient of z^k in f_n is eventually constant for large n, for each k. By a Rouché's Theorem argument, the pairs of zeros $\{z_n, \bar{z}_n\}$ of f_n must converge to (at least) a double zero at z of $\lim_{n \to \infty} f_n$. □

4. $\bar{W}$ IS CONNECTED

Since W is countable, we cannot hope to prove W is connected. We prove instead that $\bar{W}$ is connected. First we need some topological lemmas.

Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology, as usual. If $v = (v_1, v_2, \ldots, v_n)$ is a finite vector of 0's and 1's, let S_v be the set of sequences in $\{0, 1\}^\omega$ which start with v. The following lemma is the key ingredient in the connectivity proof.

LEMMA 4.1. *Let* Y *be a topological space. Suppose* $f : \{0, 1\}^\omega \to Y$ *is a continuous map such that*

$$f(S_{v0}) \cap f(S_{v1}) \neq \emptyset \tag{4.1}$$

for all $v \in \{0, 1\}^n$, *and all* $n \geqslant 0$. *(Here* $v0$ *denotes the vector* v *with* 0 *appended, etc.) Then the image of* f *is path connected.*