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ZEROS OF POLYNOMIALS WITH 0, 1 COEFFICIENTS

by A.M. ODLYZKO and B. POONEN[1]

ABSTRACT. Zeros of polynomials with 0, 1 coefficients exhibit many interesting features, including fractal appearance. This paper obtains bounds for such zeros. It shows that zeros with a sufficiently large negative real part are real. It also proves that the closure of the set of these zeros is path connected.

1. INTRODUCTION

Zeros of polynomials with random coefficients occur in many scientific and engineering problems. A general overview of the subject and references can be found in the book of Bharucha-Reid and Sambandham [4], which is the basic reference on this topic. There is a wealth of information about distribution of zeros in the complex plane and on the real line. Almost all of the results are for coefficients chosen independently from a common distribution that is continuous, and usually Gaussian.

In this paper we consider zeros of polynomials with 0, 1 coefficients. These zeros have some features that distinguish them from those of the commonly considered families of random polynomials. Let

$$(1.1) \qquad P = \left\{ f(z) : f(z) = 1 + \sum_{j=1}^{d} a_j z^j \,, \quad a_j = 0 \text{ or } 1 \text{ for all } j \right\} .$$

(We exclude polynomials with constant term 0, as their zeros, other than 0, are those of polynomials of lower degree with coefficients 0, 1.) Define

$$(1.2) \qquad W = \{ z \in \mathbf{C} : f(z) = 0 \text{ for some } f \in P \} .$$

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For each degree d, there are 2^{d-1} polynomials $f(z) \in P$ of degree d, and so W is a countable set.

There are few published results about W. In [8] it was shown that $\mathrm{Re}(z) < 3/2$ for all $z \in W$. This was used to prove that if $f(2)$ is a prime for some $f(z) \in P$, then $f(z)$ is irreducible over the rationals. (For further results relating zeros to irreducibility, see [12]. It is conjectured that almost all $f(z) \in P$ are irreducible, but this is still open. This is in contrast to the case of fixed degree polynomials when the range over which the coefficients are allowed to run increases. There it is known that almost all polynomials are not only irreducible, but also have S_n as their Galois group. For latest results and references on this topic, see [14].)

Our results are best illustrated by pictures of zeros. Figure 1 shows all zeros of the polynomials with coefficients 0, 1 of degrees $\leqslant 16$, and with constant term 1, except for the negative real zeros that are < -1.5. We show that W lies between the curves

$$C_1 = \left\{z : |z| \leqslant 1, \frac{|z|}{1 - |z|} = \left|\frac{2 - z}{1 - z}\right|\right\} \tag{1.3}$$

and

$$C_2 = \left\{z : |z| \geqslant 1, \frac{1}{|z| - 1} = \left|\frac{2z - 1}{1 - z}\right|\right\}. \tag{1.4}$$

The curve C_1 is mapped to C_2 by $z \to 1/z$. This mapping takes W to itself, since if $z \in W$, and z is a root of $f(z) \in P$ and $\deg f(z) = d$, then $1/z$ is a root of $z^d f(1/z) \in P$. We show that all $z \in W$ are enclosed strictly between C_1 and C_2. From this it follows that for all $z \in W$,

$$\frac{1}{\varphi} < |z| < \varphi, \tag{1.5}$$

where

$$\varphi = \frac{1 + 5^{1/2}}{2} \tag{1.6}$$

is the "golden ratio." (The bound (1.5) has been proved independently in different contexts by Flatto, Lagarias, and Poonen [13] and by Solomyak [16].) We also show that the line segment $[-\varphi, -\varphi^{-1}] \in \overline{W}$. However, $-\varphi \notin \overline{W}$ and $-\varphi^{-1} \notin \overline{W}$. Further, there is a constant $\delta > 0$ such that if

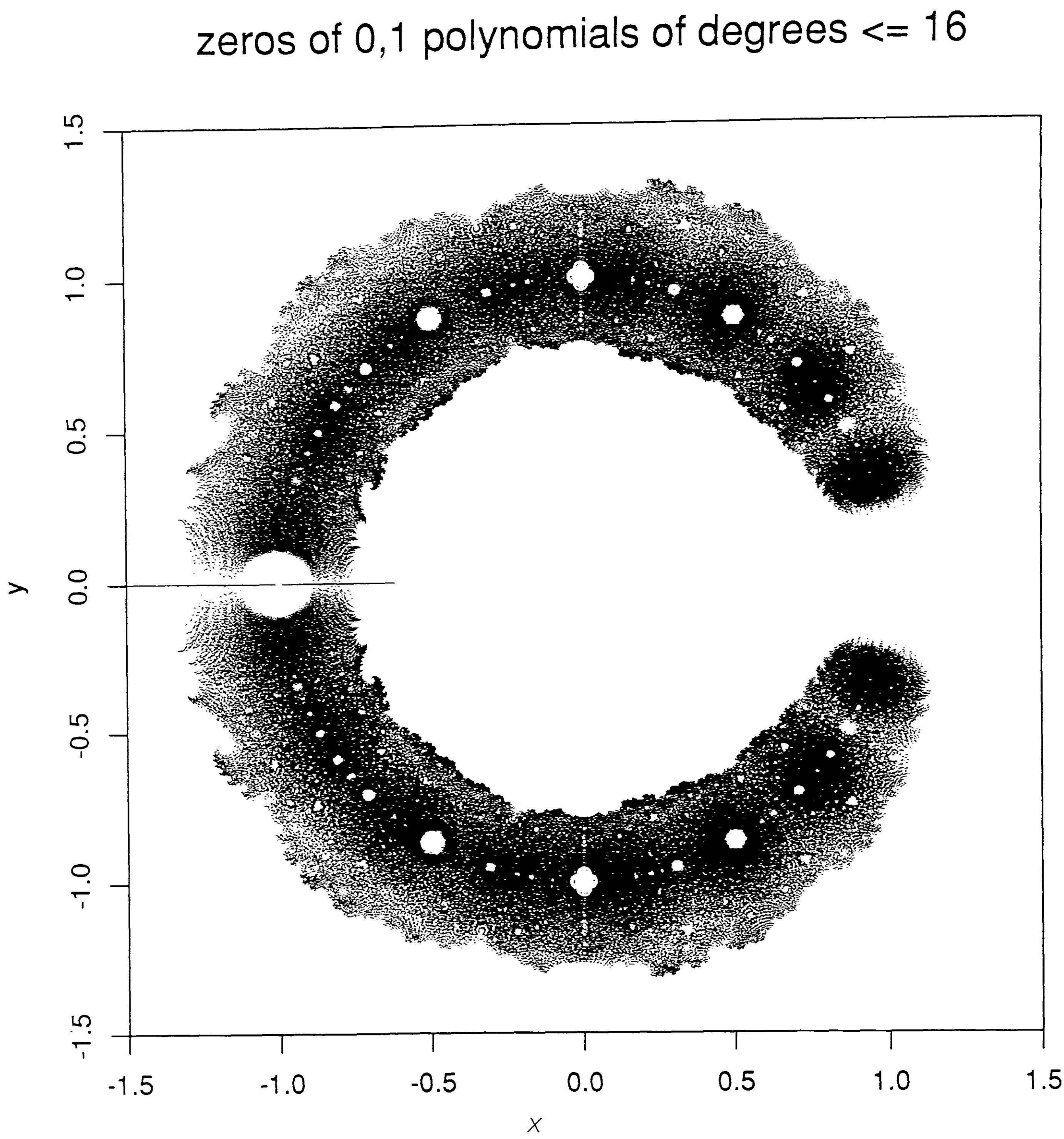

FIGURE 1

Scatterplot of zeros $z = x + iy$ of polynomials of degrees $\leqslant 16$
with constant term 1 and coefficients 0 and 1.
The "segments" along the negative real axis are created by negative real zeros.
Negative real zeros < -1.5 are not shown.

$z \in W$, $|z| \leqslant \varphi^{-1} + \delta$, then $z \in R$. Thus the "spike" along the negative real axis that is visible in Figure 1, connecting curves C_1 and C_2 with the exception of a small gap at -1, is due to zeros.

Since polynomials in P have nonnegative coefficients, $1 \notin W$. However, since $\zeta \in W$ for every root of unity $\zeta \neq 1$, $1 \in \overline{W}$, where $\overline{W}$ denotes

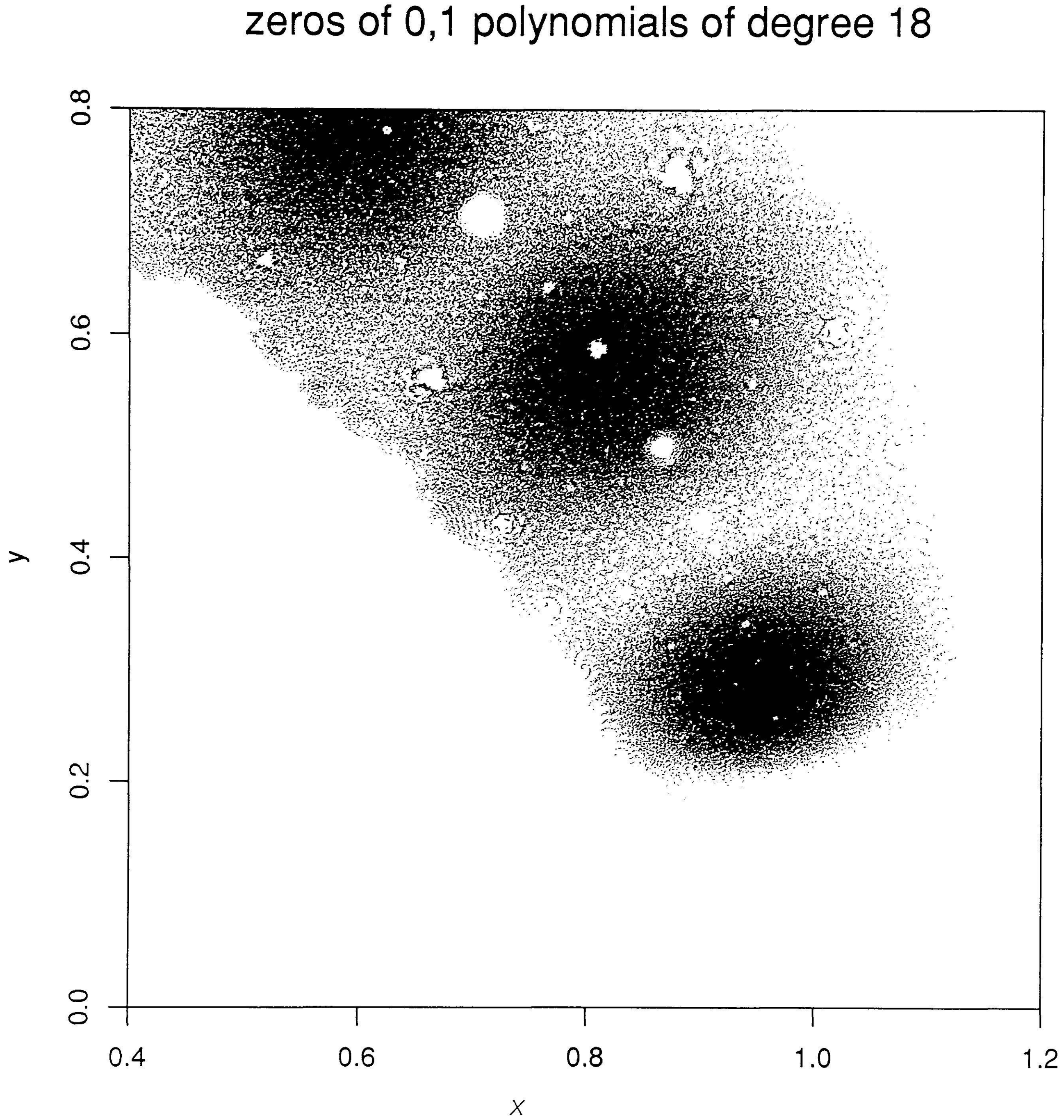

FIGURE 2

Scatterplot of zeros of polynomials of degree 18
with constant term 1 and coeffficients 0 and 1 that are near to $z = 1$.

the closure of W. We answer a question posed by J.H. Conway and Richard Parker about the behaviour of W near 1 by proving there exist points $z = x + iy \in W$ such that $0 < x - 1 y = o(|y|)$, so that these points come in tangent to the x-axis.

Figure 2 shows the zeros of polynomials $f(x) \in P$ of degree 18 that are close to $z = 1$. Figure 3 shows zeros of polynomials $f(x) \in P$ of all degrees $\leqslant 32$ that fall in a certain small region of the complex plane. Figures 4 , 5

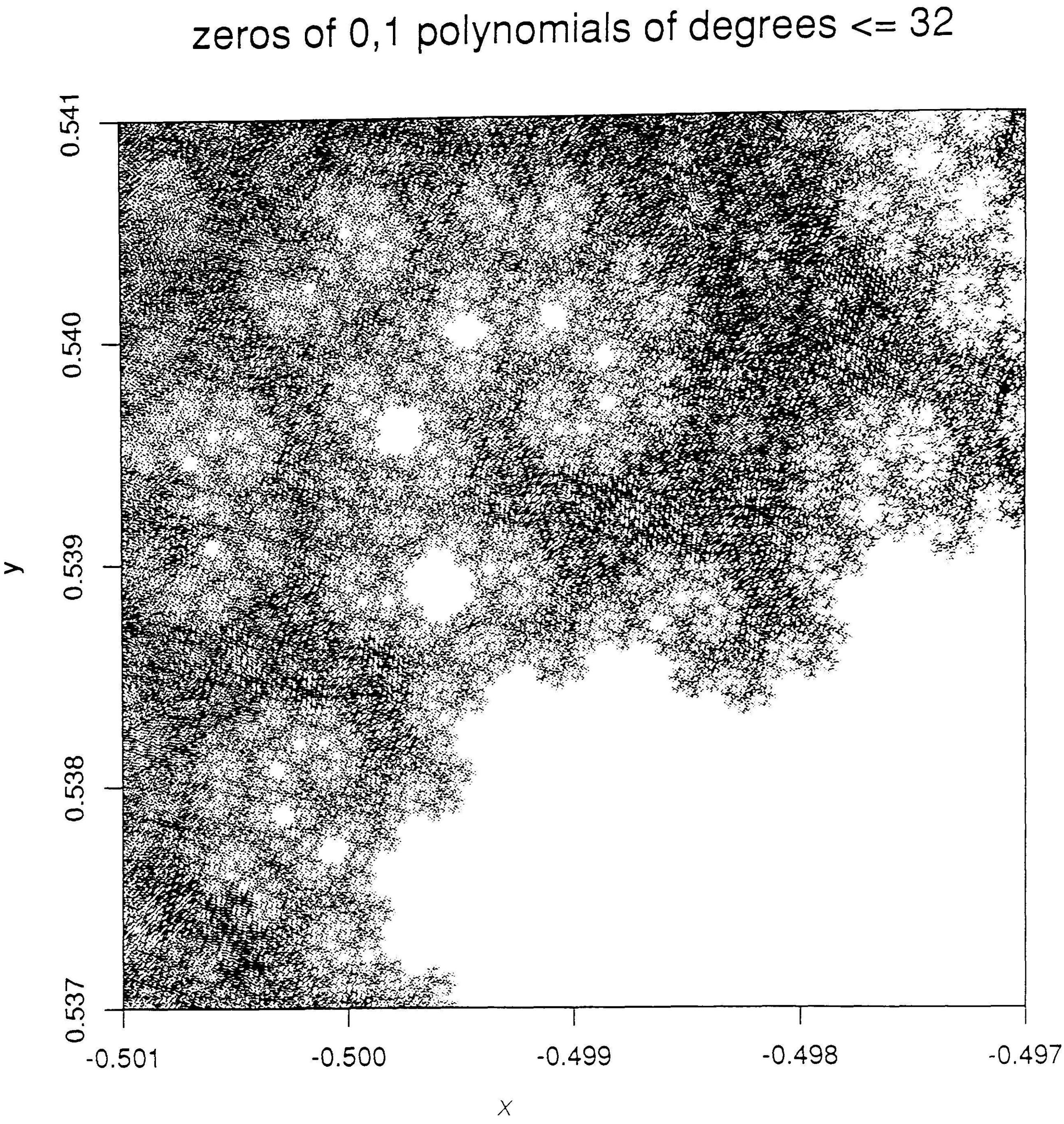

FIGURE 3

Scatterplot of zeros of polynomials of degrees $\leqslant 32$ with coefficients 0 and 1.

and 6 show pictures of parts of $\overline{W}$. The region depicted in Figure 4 is the same as that of Figure 3. Section 6 explains how these pictures were created.

Theorem 2.1 of Section 2, which says that W is contained between C_1 and C_2, is not best possible. The only points of $\overline{W}$ that are in $C_1 \cup C_2$ are $1, -\varphi, -\varphi^{-1}$. In Section 6 we will show how to obtain more precise bounds for W. However, because of the fractal nature of W, there is no simple description of its shape.

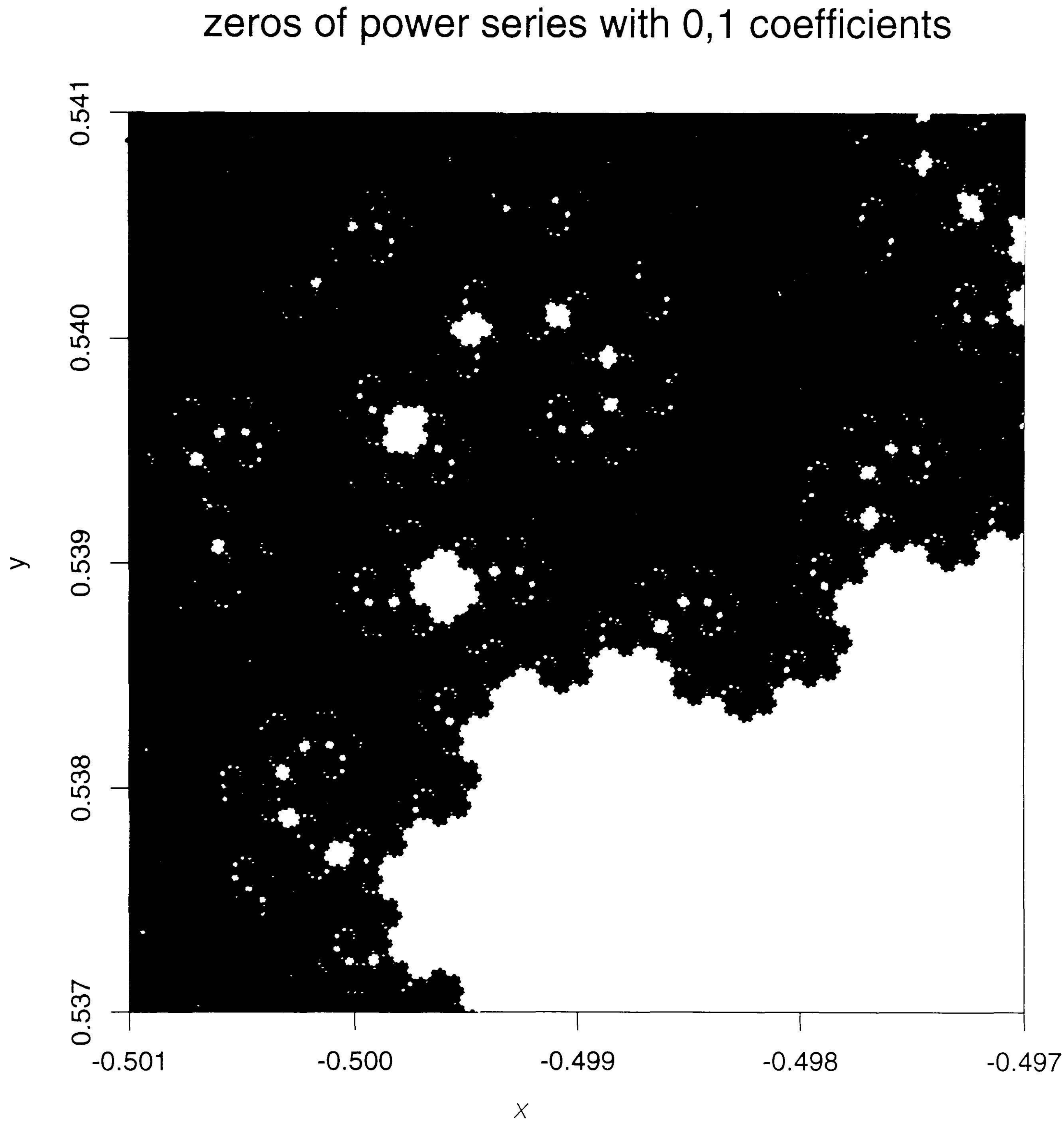

FIGURE 4

Section of $\bar{W}$. The same region, with points from W displayed, is shown in Figure 3. Black denotes points $z \in \bar{W}$.

Many features visible in the graphs can be explained (at least heuristically, and often rigorously) by using known results or methods. When one graphs zeros of any single polynomial with coefficients 0 and 1, most of them are close to the unit circle $|z| = 1$ and they are equidistributed in angles, so that the first quadrant, for example, has close to 1/4 of the total. This phenomenon is true for all polynomials whose coefficients do not vary much, as follows from results originating with Erdös and Turán [11]. For statements and references to general results, see [4].

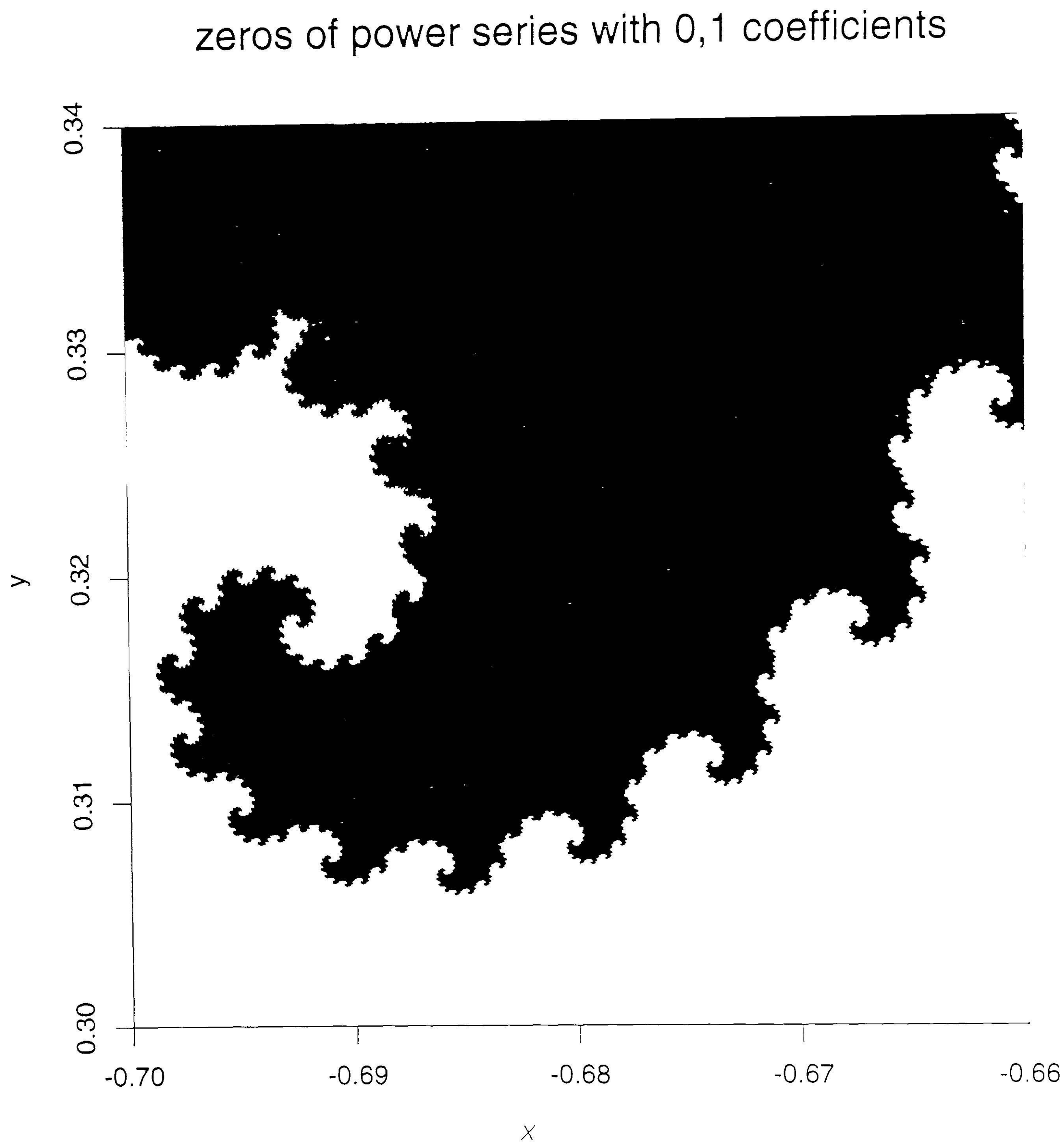

FIGURE 5

Section of $\overline{W}$, the set of zeros of power series with 0, 1 coefficients with black denoting $z \in \overline{W}$.

The expected number of real roots of a random polynomial (which have to be negative for $f(z) \in P$) grows logarithmically with n, as was first noted by Kac and Rice (see [4]). Furthermore, the variance is small.

In Figures 1 and 2, there is a perceptible clustering of zeros. This is a reflection of the "averaging phenomenon" for roots of random polynomials [4, 15], and again is not special to 0, 1 coefficients. The "average" of the polynomials of degree n that are in P is

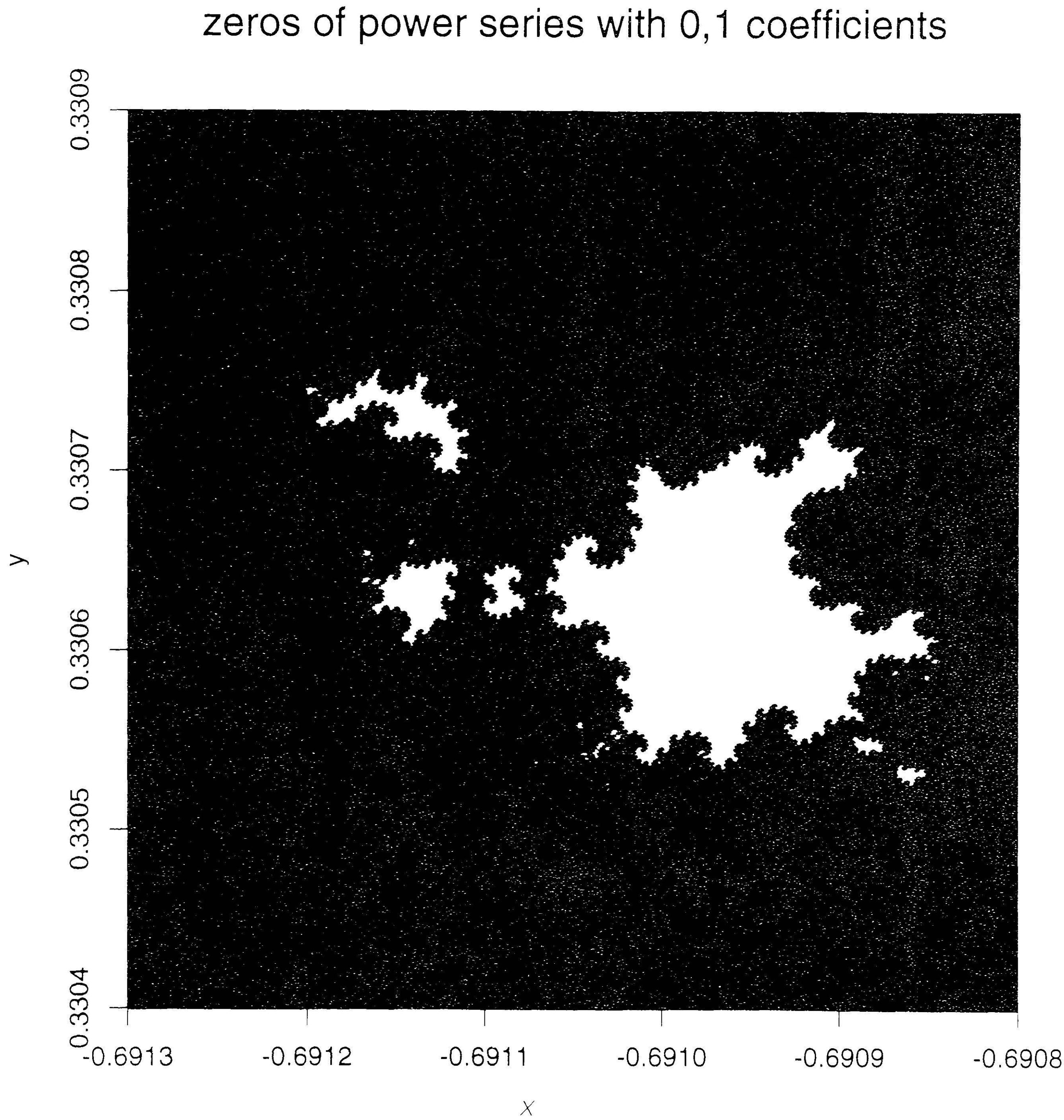

FIGURE 6

Section of $\bar{W}$. This is an enlargement by a factor of 80 of a section of Figure 5, showing some of the holes contained in $\bar{W}$.

$$g(z) = z^n + 1 + \frac{1}{2}\sum_{k=1}^{n-1} z^k = \frac{(1-2z)z^n - z + 2}{2(1-z)}, \tag{1.7}$$

and on average the zeros of $f(z) \in P$ tend to cluster near the zeros of $g(z)$.

Figures 1 and 2 show several large "holes," which contain either just one or no zeros. These holes are usually centered at algebraic integers α of low degree and small height (i.e., algebraic integers α that satisfy polynomial equations with small integral coefficients). The most prominent of the holes are at the roots of unity, such as -1 and i. As one computes zeros of polynomials $f(z) \in P$ of increasing degrees, the large holes in Figures 1 and 2 fill up. However, there are other holes, such as those visible in Figures 3-6, that are free of zeros even when the degree increases.

We show in Section 3 that there is an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ that is in $\bar{W}$. In Section 4 we prove that $\bar{W}$ is connected. The more involved argument in Section 5 proves that $\bar{W}$ is path connected. Since the unit circle is contained in $\bar{W}$, but $0 \notin \bar{W}$, $\bar{W}$ is clearly not simply connected. Numerical experiments suggest that $\bar{W}$ has "holes" in it besides the big hole containing 0. (That is, $\mathbf{C} \setminus \bar{W}$ has more than 2 connected components.) In particular, the disk of radius 10^{-5} centered at $-0.69098 + 0.33062i$ appears to be part of such a hole. This hole and some neighboring ones are pictured in Figures 5 and 6. Other, even larger holes, can be seen in Figures 3 and 4.

W has a fractal appearance that is reminiscent of some of the Julia sets [1, 10]. In Section 6 we sketch arguments that explain how this arises. However, we do not have estimates for such interesting parameters as the Hausdorff dimension of the boundary of $\bar{W}$.

In contrast to our result that $\bar{W}$ is path connected, the Mandelbrot set is only known to be connected, although it is conjectured to be path connected [1, 10]. Our methods are simpler than those used to study the connectedness of the Mandelbrot set. They are similar to the techniques developed for investigating iterated function systems [1].

Results similar to those for polynomials with 0, 1 coefficients can also be obtained for other families of polynomials with a small set of possible coefficients. For example, for ± 1 coefficients, pictures of zeros are qualitatively similar to those of 0, 1 polynomials. There is symmetry about the imaginary axis as well as the real axis (corresponding to changing the variable z to $-z$). There are two "spikes" of zeros along the real axis that fill the intervals $[-2, -1/2]$ and $[1/2, 2]$, while there are no other zeros in $|z| \leqslant 1/2 + \delta$ or $|z| \geqslant 2 - \delta$ for some $\delta > 0$. For polynomials with cubic roots of unity as coefficients, there are no "spikes", but the zeros still have a fractal appearance.

The set

$$\bar{W} \cap \{z : |z| < 1\} \tag{1.8}$$

is the set of zeros of power series

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \qquad a_k = 0 \text{ or } 1 . \tag{1.9}$$

Since $z^{-1} \in W$ for all $z \in W$, it is sufficient to study $z \in W, |z| \leqslant 1$, and in some ways it is more natural to deal with the above power series.

Some of our methods and results are similar to those of Thierry Bousch [5, 6], whose work was brought to our attention by D. Zagier. The report [5] proves that the closure of the set of zeros of polynomials with coefficients 0, ± 1 is connected. The thesis [6] contains, along with a variety of other results, general methods for studying similar problems. In the area where our work overlaps [5, 6], we obtain a somewhat stronger result by proving path connectivity.

Boris Solomyak [16] has studied zeros of power series of the form (1.9), but with the c_k, $k \geqslant 1$, allowed to take any real values in the interval [0, 1]. He shows that the bound (2.4) holds there as well, and that there is a "spike" of real zeros along the negative real axis. However, the zeros of Solomyak's functions are substantially different from those we investigate. For example, he shows that segments of the boundary he investigates have everywhere dense sets of points where a tangent exists, as well as everywhere dense sets of points with no tangent. There are also no holes in Solomyak's set of zeros.

The paper of Brenti, Royle, and Wagner [7] discusses various properties of chromatic polynomials. While it is not directly related to our work, the numerical evidence it presents shows that zeros of chromatic polynomials may also exhibit fractal behavior. This may also be true for the partition function zeros of [3].

2. Bounds and Locations for Zeros

A polynomial $f(z) \in P$ can have multiple zeros. If $\zeta \neq 1$ is a d-th root of unity, then ζ is a zero of

$$g(z) = \sum_{j=0}^{d-1} z^j ,$$

and therefore a zero of $g(z^k)$ for any k such that $d \mid k - 1$. Hence it is a zero of multiplicity 2 for $g(z) g(z^k)$, a polynomial in P. Higher multiplicities can be obtained by iterating this procedure. On the other hand, we do not know whether any $z \in W$ that is not a root of unity can be a multiple root of any $f(z) \in P$. There do exist power series with coefficients 0, 1 that have double zeros z with $|z| < 1$, as will be shown in Section 3.

Inside a disk $\{z : |z| < r\}$ for $r < 1$, any polynomial $f(z) \in P$ can have only a bounded number of zeros. We prove a slightly more general result that will be used later on.

PROPOSITION 2.1. *Suppose that* $f(z)$ *is a power series of the form*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \qquad a_k = 0, 1 .$$

Then for any $r, 0 < r < 1, f(z)$ *has*

$$\leqslant 2(-\log(1 - r^{1/2}))(-\log r)^{-1} \tag{2.1}$$

zeros in $|z| \leqslant r$.

Proof. We apply Jensen's theorem (Theorem 3.61 of [17]). If $z_1, \ldots, z_n$ are the zeros in $|z| < R$, where $r < R < 1$, then we find that

$$\sum_{j=1}^{n} \log(R/|z_j|) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta , \tag{2.2}$$

since $f(0) = 1$. Therefore, if m is the number of zeros in $|z| < r$, we have

$$m(\log R - \log r) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta . \tag{2.3}$$

Since

$$|f(Re^{i\theta})| \leqslant \sum_{k=0}^{\infty} R^k = (1 - R)^{-1} ,$$

we obtain

$$m \leqslant (-\log(1 - R))(\log R - \log r)^{-1} .$$

We now choose $R = r^{1/2}$, and this yields the bound (2.1). (Better bounds can be obtained by selecting R more carefully or estimating the integral of $\log|f(z)|$ in Jensen's theorem better.) □

We next consider bounds on the size of $z \in W$. Since $1/z \in W$ for $z \in W$, it suffices to consider $|z| \leqslant 1$.

THEOREM 2.1. *Suppose that* z *satisfies* $|z| < 1$ *and that* $f(z) = 0$ *for some power series of the form* (1.9). *Then*

$$\frac{|z|}{1 - |z|} \geqslant \left| \frac{2 - z}{1 - z} \right| , \tag{2.4}$$

and $|z| \geqslant \varphi^{-1}$, *with equality if and only if* $z = -\varphi^{-1}$ *and* $f(z) = 1 + z + z^3 + z^5 + \cdots$. *Furthermore, there exists a* $\delta > 0$ *such that if* $|z| < \varphi^{-1} + \delta$, *then* z *is a negative real number.*

Proof. We note that

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k = 1 + \frac{1}{2} \sum_{k=1}^{\infty} z^k + g(z)$$

$$(2.5) \qquad = \frac{2 - z}{2(1 - z)} + g(z)$$

where

$$(2.6) \qquad g(z) = \frac{1}{2} \sum_{k=1}^{\infty} \varepsilon_k z^k , \qquad \varepsilon_k = \pm 1 \text{ for all } k .$$

Now for $|z| < 1$,

$$(2.7) \qquad |g(z)| \leqslant \frac{|z|}{2(1 - |z|)} ,$$

and so we conclude that if (2.4) is violated, then $f(z) \neq 0$. Equality is possible in (2.7) only if z is real, and we easily check that for $z \in (-1, 1)$,

$$\frac{|z|}{1 - |z|} = \left|\frac{2 - z}{1 - z}\right|$$

only at $z = -\varphi^{-1}$. For $z = -\varphi^{-1}$, equality holds in (2.7) when $\varepsilon_k = (-1)^{k-1}$ for $k \geqslant 1$; i.e., when

$$f(x) = 1 + x + x^3 + x^5 \cdots .$$

Let $\Delta = \{z : |z| < \varphi^{-1}\}$. Then $z \mapsto (2 - z)/(1 - z)$ maps Δ to the interior of the circle one of whose diameters is

$$\left[\frac{2 + \varphi^{-1}}{1 + \varphi^{-1}}, \frac{2 - \varphi^{-1}}{1 - \varphi^{-1}}\right] = [\varphi, 2 + \varphi]$$

but $z \mapsto |z|/(1 - |z|)$ maps Δ to

$$\left[0, \frac{\varphi^{-1}}{1 - \varphi^{-1}}\right) = [0, \varphi)$$

so (2.4) fails if $z \in \Delta$. Moreover, if $z \in \partial\Delta$, (2.4) still fails unless

$$\frac{2 - z}{1 - z} = \varphi$$

$$z = \frac{\varphi - 2}{\varphi - 1} = -\varphi^{-1} .$$

We next prove that the only $z \in \bar{W}$ with $|z|$ close to φ^{-1} are negative real numbers. Since $\bar{\Delta}$ intersects the closed set

$$\left\{z : |z| \leqslant 0.9 \text{ and } \frac{|z|}{1-|z|} \geqslant \left|\frac{2-z}{1-z}\right|\right\}$$

only at $z = -\varphi^{-1}$, there exist $\delta_1, \delta_2 \in (0, 10^{-10})$ such that (2.4) fails for z in

(2.8) $$S = \{z : |z| \leqslant \varphi^{-1} + \delta_1, |z + \varphi^{-1}| \geqslant \delta_2\} .$$

It only remains to find the possible elements of $\bar{W}$ that lie in

(2.9) $$T = \{z : |z + \varphi^{-1}| < \delta_2, |z| < \varphi^{-1} + \delta_1\} .$$

For $z \in T$,

(2.10) $$\operatorname{Re}\left(1 + \frac{z}{2(1-z)}\right) \geqslant \varphi^{-1} - 10|z + \varphi^{-1}| \geqslant \varphi^{-1} - 10^{-9} ,$$

so if $z \in W$, then we must have $\operatorname{Re} g(z) \leqslant -\varphi^{-1} + 10^{-9}$. Since $|g'(z)| \leqslant 10$ for $z \in T$, and $|g(-\varphi^{-1})| \leqslant \varphi^{-1}$, to achieve $\operatorname{Re} g(z) \leqslant -\varphi^{-1} + 10^{-9}$, we must have $\varepsilon_k = (-1)^{k-1}$ for $1 \leqslant k \leqslant 20$, say. Then

(2.11) $$f(z) = 1 + z + z^3 + \cdots + z^{19} + h(z) ,$$

where

(2.12) $$h(z) = \sum_{k=21}^{\infty} a_k z^k , \qquad a_k = 0 \text{ or } 1 .$$

Hence for $|z| = r$,

$$|h(z)| \leqslant \frac{r^{21}}{1-r} ,$$

while

(2.13) $$1 + z + z^3 + \cdots + z^{19} = 1 + z\frac{1 - z^{20}}{1 - z^2} = \frac{1 + z - z^2 - z^{21}}{1 - z^2} .$$

On the circle $|z| = r = 0.7$,

$$|1 + z - z^2| = |z - \varphi||z + \varphi^{-1}| \geqslant 0.08 \cdot 0.9 \geqslant 0.07 ,$$

so

(2.14) $$\left|\frac{1 + z - z^2 - z^{21}}{1 - z^2}\right| \geqslant \frac{0.07 - (0.7)^{21}}{1 + (0.7)^2} \geqslant 0.03 .$$

On the other hand, $(0.7)^{21}/0.3 < 0.01$, so by Rouché's theorem $f(z)$ and $(1 + z - z^2 - z^{21})/(1 - z^2)$ have the same number of zeros inside $|z| \leqslant 0.7$. By the earlier part of the argument, and another application of Rouché's theorem, $1 + z - z^2$ and $1 + z - z^2 - z^{21}$ have the same number of zeros inside $|z| \leqslant 0.7$, namely one. Therefore $f(z)$ has exactly one zero inside $|z| \leqslant 0.7$, and since $f(z)$ has real coefficients, this zero has to be real. □

The argument presented above is inefficient, and shows only that some value of $\delta < 10^{-10}$ is allowable. With a little more care one could show by an extension of the method used above that $\delta = 0.7 - \varphi^{-1} = 0.081\ldots$ is allowable, so that any $z \in W$ with $|z| < 0.7$ is real. In Section 6 we present a variation of this method that uses machine computation instead of careful estimates to establish rigorously that $\delta = 0.7 - \varphi^{-1}$ is allowable. Numerical evidence suggests that the minimal value of $|z|$ over $z \in \bar{W}\backslash\mathbf{R}$ is about 0.734. The method of Section 6 can be used to obtain estimates for the minimal value of $|z|$ over $z \in \bar{W}\backslash\mathbf{R}$ that are as accurate as one desires.

By Proposition 3.1 of the next section, $(-1, -\varphi^{-1}] \subseteq \bar{W}$. Since $\bar{W}$ is stable under $z \mapsto 1/z$ and closed, it follows that $[-\varphi, -\varphi^{-1}] \subseteq \bar{W}$.

In [8] it was shown that $z \in W$ implies $\operatorname{Re}(z) < 3/2$. Theorem 2.1 immediately leads to the bound $\operatorname{Re}(z) < 1.22$ for $z \in W$. Numerical evidence suggests that $\operatorname{Re}(z) < 1.14$ for $z \in W$. There are $z \in W$ with $\operatorname{Re}(z) > 1.13$. The methods outlined in Section 6 can be used to obtain precise bounds.

We can analyze inequality (2.4) for z close to 1. We find that for $z = 1 - x + iy$ with x and y small, $x > 0$, if $|y| \leqslant x^{3/2}$ then (2.4) fails, so $z \notin W$. We next show that there are points in W which approach 1 along trajectories tangent to the real axis.

PROPOSITION 2.2. *There exists a sequence of points* $z_n \in W$ *such that* $z_n \to 1$ *as* $n \to \infty$ *and*

$$|\operatorname{Im}(z_n - 1)| = o(\operatorname{Re}(z_n - 1)) \quad \text{as} \quad n \to \infty. \tag{2.15}$$

Proof. Consider the polynomial

$$f_{m,n}(z) = 1 + z + \cdots + z^{m-1} + z^n \tag{2.16}$$

with $m \leqslant n$. For n large compared to m, we will show that $f_{m,n}(z)$ has a zero near to

$$\alpha = \alpha_{m,n} = \exp(\pi i/n + (\log m)/n), \tag{2.17}$$

and $\operatorname{Re}(\alpha - 1) \sim (\log m)\operatorname{Im}(\alpha)$. We show that one can take $m \lesssim n/(\log n)$.

To show that $f_{m,n}(z)$ has a zero β near $\alpha = \alpha_{m,n}$, let

$$g(z) = m + z^n . \tag{2.18}$$

Then $g(\alpha) = 0$. Consider the circle $|z - \alpha| = (10n)^{-1}$. On this circle, $|g(z)| \geqslant m/100$, while

$$|(1 + z + \cdots + z^{m-1}) - m| \leqslant \sum_{k=1}^{m-1} |z^k - 1| = O(m^2/n) , \tag{2.19}$$

so for $m = o(n)$, by Rouché's theorem $g(z)$ and $f_{m,n}(z)$ have the same number of zeros inside the circle, namely one. This proves the claim and answers the Conway-Parker question. □

3. A NEIGHBORHOOD OF THE UNIT CIRCLE

In this section we prove that an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ is contained in $\bar{W}$.

LEMMA 3.1. *If* $B \subseteq \mathbf{C}$ *is compact,* $n \geqslant 1, |z| < 1,$ *and*

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0,1\}} \left[\left(\sum_{i=1}^{n} \varepsilon_i z^i\right) + z^n B\right] , \tag{3.1}$$

then every element of B *is expressible in the form*

$$\sum_{i=1}^{\infty} \varepsilon_i z^i, \qquad \varepsilon_i \in \{0, 1\} . \tag{3.2}$$

In particular, if $-1 \in B$, *then* $z \in \bar{W}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $\varepsilon_{mi} \in \{0, 1\}$, $m \geqslant 0$, $1 \leqslant i \leqslant n$ such that

$$b_m = \left(\sum_{i=1}^{n} \varepsilon_{mi} z^i\right) + z^n b_{m+1} .$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^{n} \varepsilon_{mi} z^{mn+i}\right) + z^{Mn} b_M .$$

Since B is compact, $z^{Mn} b_M \to 0$ as $M \to \infty$, so

$$b_0 = \sum_{m=0}^{\infty} \sum_{i=1}^{n} \varepsilon_{mi} z^{mn+i} ,$$

which is the desired form. □

PROPOSITION 3.1. *If* $z \in \mathbf{R}$, $-1 < z \leqslant -\varphi^{-1}$, *then* $z \in \overline{W}$.

Proof. Let $B = [-1, -z]$. Then, since $-1 < z \leqslant -\varphi^{-1}$ implies $z - z^2 \leqslant -1$, we have

$$\begin{aligned}(z + zB) \cup zB &= [z - z^2, 0] \cup [-z^2, -z] \\ &= [z - z^2, -z] \\ &\supseteq [-1, -z] \\ &= B .\end{aligned}$$

We now apply Lemma 3.1 with $n = 1$, and conclude that $z \in \overline{W}$. □

LEMMA 3.2. *If* $B \subseteq \mathbf{C}$ *is compact,* $-1 \in B$, $n \geqslant 1$, $x \in \mathbf{C}$ *and*

$$B \subseteq \operatorname{int} \bigcup_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} \left[\left(\sum_{i=1}^{n} \varepsilon_i x^i \right) + x^n B \right] , \tag{3.3}$$

where $\operatorname{int} S$ *denotes the interior of* S, *then there is a neighborhood* N *of* x *such that*

$$N \cap \{z : |z| < 1\} \subseteq \overline{W} .$$

Proof. Condition (3.3) implies that (3.1) holds for z in a neighborhood of x, so Lemma 3.2 follows from Lemma 3.1. □

LEMMA 3.3. *If* $B = \{z : |z| \leqslant R\}$ *for some* $R \geqslant 1$, $n \geqslant 1$, $|x| = 1$ *and*

$$B \subseteq \operatorname{int} \bigcup_{j=1}^{n} (x^j + B) , \tag{3.4}$$

then

$$x \in \operatorname{int} \overline{W} .$$

Proof. Since $\overline{W}$ contains the unit circle and is closed under $z \mapsto 1/z$, this follows trivially from Lemma 3.2. □

PROPOSITION 3.2. *If* $|x| = 1$, $x \neq \pm 1$, *then* $x \in \operatorname{int} \overline{W}$.

Proof. We claim that if $R \geqslant 2$, then the condition (3.4) of Lemma 3.3 holds for n large enough. If $x = \exp(\pi i \alpha)$ and α is irrational, then by Kronecker's theorem $\{x^j : j \geqslant 1\}$ is dense on the unit circle, and then for every $\delta > 0$, the disk of radius $R + 1 - \delta$ is contained in the union on the right side of (3.4) for n large enough. If α is rational, then the $\{x^j : j \geqslant 1\}$ are the vertices of a regular k-gon, and $k \geqslant 3$ since $x \neq \pm 1$. In that case the union on the right side of (3.4) contains a disk of radius r, where r, 1, and R are the sides of a triangle, and the angle between the sides of lengths r and 1 is π/k. Therefore, by the Law of Cosines,

$$R^2 = 1 + r^2 - 2r\cos(\pi/k) ,$$

and so

$$r = \cos(\pi/k) + (\cos^2(\pi/k) + R^2 - 1)^{1/2} .$$

Since $\cos(\pi/k) \geqslant \cos(\pi/3)$, we find that

$$r \geqslant 1/4 + (R^2 - 3/4)^{1/2} \geqslant R + 1/20$$

for $R \geqslant 2$, since $(R^2 - 3/4)^{1/2} - R$ is an increasing function of R. $\square$

Proving $-1 \in \operatorname{int} \overline{W}$ is trickier, because it will not do to take B as a disc of radius $\geqslant 1$ if $\operatorname{Im}(z)$ is small compared to $\operatorname{Re}(z+1)$. We will instead take B as a parallelogram that becomes flatter and flatter as $\operatorname{Im} z \to 0$. The following two lemmas will be used in verifying the condition of Lemma 3.1.

LEMMA 3.4. *Let* $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. *Let* $v_j = T^j\binom{1}{0} = (-1)^j\binom{1}{-j}$. *Then for* $n \geqslant 16$

$$\left\{\sum_{j=1}^{n} \varepsilon_j v_j : \varepsilon_j \in \{0, 1\}\right\}$$

contains $\{\binom{a}{b} : a, b \in \mathbf{Z}, |a| \leqslant 1, |b| \leqslant n - 16\}$.

Proof. Given such $\binom{a}{b}$, first pick ε_1, ε_2 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2$ has first coordinate a. Next pick $\varepsilon_3 = \varepsilon_4 = 0$ or 1 so that $\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4$ has first coordinate a and second coordinate b' with $b' \not\equiv b \bmod 2$. Certainly $|b'| \leqslant 1 + 2 + 3 + 4 = 10$, so $|b - b'| \leqslant n - 6$. If $b > b'$, then

$$\begin{aligned} &\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_5 + v_{5+b-b'} \\ &= (a, b') - (1, -5) + (1, 5 + b - b') \\ &= (a, b) . \end{aligned}$$

If $b < b'$, then

$$\begin{aligned} &\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3 + \varepsilon_4 v_4 + v_6 + v_{6+b'-b} \\ &\quad = (a, b') + (1, -6) - (1, 6 + b' - b) \\ &\quad = (a, b) . \qquad \square \end{aligned}$$

LEMMA 3.5. *Let* T, v_j *be as in Lemma 3.4. Let* B *be the square with vertices* $(\pm 1, \pm 1)$. *Then for* $n \geqslant 35$,

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0, 1\}} \left[\left(\sum_{j=1}^{n} \varepsilon_j v_j \right) + \frac{1}{2} T^n B \right] .$$

Proof. $\frac{1}{2} T^n$ is the parallelogram with vertices

$$\pm \frac{1}{2} (1, -n) \pm \frac{1}{2} (0, 1) .$$

The cross-section of this with x-coordinate x_0 is the vertical interval $[-nx_0 - 1/2, -nx_0 + 1/2]$ for $-1/2 \leqslant x_0 \leqslant 1/2$. Hence given $(\alpha, \beta) \in B$ pick $a \in \{-1, 0, 1\}$ such that $-1/2 \leqslant \alpha + a \leqslant 1/2$ and then pick $b \in \mathbf{Z}$ such that $-n(\alpha + a) + 1/2 \leqslant \beta + b \leqslant -n(\alpha + a) + 1/2$. Since $|\beta| \leqslant 1$ and $|\alpha + a| \leqslant 1/2$, we see $|b| \leqslant \frac{1}{2}(n + 1) + 1 \leqslant n - 16$ if $n \geqslant 35$. Then $(\alpha, \beta) + (a, b) \in \frac{1}{2} T^n B$ and by Lemma 3.4 we can pick $\varepsilon_1, \ldots, \varepsilon_n$ such that $\sum_{j=1}^{n} \varepsilon_j v_j = -(a, b)$, so Lemma 3.5 follows. $\square$

PROPOSITION 3.3. $-1 \in \operatorname{int} \bar{W}$.

Proof. Since $\bar{W}$ is closed under $z \mapsto \bar{z}$ and $z \mapsto 1/z$, it suffices to show that for $|z| < 1$, $\operatorname{Im} z > 0$, and $|z + 1|$ sufficiently small, z is in $\bar{W}$. (Proposition 3.1 handles the case $z \in \mathbf{R}$.) Let $\delta = z + 1$. Let B be the parallelogram with vertices $\pm 1 \pm \delta$.

We work in a nonstandard coordinate system for $\mathbf{C}$, with basis vectors 1 and δ, so B is represented by the square with vertices $(\pm 1, \pm 1)$. We claim that multiplication by z is represented by the matrix $T = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ up to $O(|\delta|)$. We have

$$\begin{aligned} z \cdot 1 &= -1 + \delta \\ z \cdot \delta &= -\delta + \delta^2 \end{aligned}$$

and

$$\delta^2 - 2(\operatorname{Re} \delta)\delta + |\delta|^2 = 0$$

so δ^2 corresponds to $(|\delta|^2, -2 \operatorname{Re} \delta)$ in our basis, and is $O(|\delta|)$.

From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \dots, \varepsilon_{35} \in \{0,1\}} \left[\left(\sum_{j=1}^{n} \varepsilon_j z^j \right) + \left(\frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small δ, we may apply Lemma 3.1 to deduce $z \in \bar{W}$. $\square$

We now combine all the results of this section.

THEOREM 3.1. *There is an open neighborhood of* $\{z : |z| = 1, z \neq 1\}$ *contained in* $\bar{W}$.

Proof. Apply Propositions 3.2 and 3.3. $\square$

COROLLARY 3.1. *If* $z \in (-1, -1+\delta)$ *for sufficiently small* δ *then* z *is a multiple zero of some* 0, 1 *power series.*

Proof. By Theorem 3.1, if δ is small enough we can pick 0, 1 power series f_n and zeros z_n of f_n such that $z_n \notin \mathbf{R}$ and $z_n \to z$ as $n \to \infty$. By taking a subsequence we may assume that the coefficient of z^k in f_n is eventually constant for large n, for each k. By a Rouché's Theorem argument, the pairs of zeros $\{z_n, \bar{z}_n\}$ of f_n must converge to (at least) a double zero at z of $\lim_{n \to \infty} f_n$. $\square$

4. $\bar{W}$ IS CONNECTED

Since W is countable, we cannot hope to prove W is connected. We prove instead that $\bar{W}$ is connected. First we need some topological lemmas.

Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology, as usual. If $v = (v_1, v_2, \dots, v_n)$ is a finite vector of 0's and 1's, let S_v be the set of sequences in $\{0, 1\}^\omega$ which start with v. The following lemma is the key ingredient in the connectivity proof.

LEMMA 4.1. *Let* Y *be a topological space. Suppose* $f : \{0, 1\}^\omega \to Y$ *is a continuous map such that*

$$f(S_{v0}) \cap f(S_{v1}) \neq \emptyset \tag{4.1}$$

for all $v \in \{0, 1\}^n$, *and all* $n \geq 0$. *(Here* $v0$ *denotes the vector* v *with* 0 *appended, etc.) Then the image of* f *is path connected.*

Proof. Let $w(0) = f(x_0')$ and $w(1) = f(x_1)$ be elements of the image we wish to connect by a path. Find $x_{1/2}, x_{1/2}' \in \{0, 1\}^\omega$ such that $x_0', x_{1/2}$ have the same first coordinate, and $x_{1/2}', x_1$ have the same first coordinate and $f(x_{1/2}) = f(x_{1/2}')$. (If x_0', x_1 have the same first coordinate, take $x_{1/2} = x_{1/2}' = x_0'$; otherwise apply the hypothesis (4.1) with υ as the empty vector.) Let $w(1/2)$ be this common value.

Next find $x_{1/4}, x_{1/4}' \in \{0, 1\}^\omega$, using the same argument, such that $x_0', x_{1/4}$ agree in the first two coordinates, $x_{1/4}', x_{1/2}$ agree in the first two coordinates, and $f(x_{1/4}) = f(x_{1/4}')$. Let $w(1/4)$ be this common value. Do the analogous thing at 3/4.

By induction, we may continue to define $x_{d/2^n}, x_{d/2^n}', w(d/2^n)$ at all dyadic rationals $d/2^n$ in $[0, 1]$, such that $x_{d/2^n}'$ and $x_{(d+1)/2^n}$ agree in the first n coordinates and

$$w(d/2^n) = f(x_{d/2^n}) = f(x_{d/2^n}') .$$

By induction, we see that all the x_q' with $q \in [d/2^n, (d+1)/2^n)$ agree in the first n coordinates. Hence for

$$r = \sum_{i=1}^{\infty} \varepsilon_i 2^{-i} \in [0, 1], \qquad \varepsilon_i \in \{0, 1\}$$

not a dyadic rational, we may define

$$x_r = x_r' = \lim_{n \to \infty} x_{\sigma(n)}' \qquad \text{where} \qquad \sigma(n) = \sum_{i=1}^{n} \varepsilon_i 2^{-i} ,$$

and $w(r) = f(x_r)$. Then w maps $[d/2^n, (d+1)/2^n]$ into $f(S_\upsilon)$ where $\upsilon \in \{0, 1\}^n$ is the first n coordinates of x_r', $r \in [d/2^n, (d+1)/2^n)$ and of $x_{(d+1)/2^n}$.

We now show that w is continuous at $r \in [0, 1]$. Let U be an open set of Y containing $w(r)$. Then $f^{-1}(U)$ contains S_υ and $S_{\upsilon'}$ for some finite substrings υ, υ' of x_r, x_r' respectively, by continuity of f. By the last sentence of the previous paragraph it follows that

$$w^{-1}(U) \supseteq w^{-1}(f(S_\upsilon) \cup f(S_{\upsilon'}))$$

will contain a neighborhood of r.

Thus $w\colon [0, 1] \to \text{image}\,(f)$ is a continuous path, and image (f) is path connected. $\square$

Let M be a topological space. Give M^n the product topology and let the symmetric group S_n act on M^n by permuting the coordinates. The space

M^n/S_n, which parameterizes n-element multisets, can be given the quotient topology.

LEMMA 4.2. *If* $A \subseteq M^n/S_n$ *is connected, and the multiset* $\{P, P, \ldots, P\}$ *is in* A *for some* $P \in M$, *then the subset* $B \subseteq M$ *of all coordinates of points in* A *is connected.*

Proof. Suppose not. Then there are open sets $U, V \subseteq M$ such that $U \cap B$ and $V \cap B$ are disjoint nonempty sets with union B. Without loss of generality, $P \in U$. Let

$$U' = U \times U \times \cdots \times U,$$

$$\begin{aligned} V' = &(V \times M \times M \times \cdots \times M) \\ &\cup (M \times V \times M \times \cdots \times M) \\ &\vdots \\ &\cup (M \times M \times M \times \cdots \times V). \end{aligned}$$

Then U', V' are open sets in M^n which are stable under S_n, so they project to open sets U'', V'' in M^n/S_n. Also $A \subseteq U'' \cup V''$ since a point in A must have all coordinates in U, or else at least one coordinate in $B \backslash U \subseteq V$. Furthermore $P \in U'' \cap A$, and $V'' \cap A$ is nonempty also, since at least one point of A has a coordinate in V, since $V \cap B \neq \emptyset$. Finally $U'' \cap V'' \cap A = \emptyset$, since it is not possible for a point of A to have all coordinates in U, yet have some coordinate in V. This contradicts the connectedness of A. □

THEOREM 4.1. $\bar{W}$ *is connected.*

Proof. First we show that for $\delta \in (0, 1)$,

$$\bar{W}_\delta = (\bar{W} \cap \{z : |z| \leqslant 1\}) \cup \{z : 1 - \delta \leqslant |z| \leqslant 1\}$$

is connected. The idea is to apply Lemma 4.1 to the function f which assigns to $(\varepsilon_1, \varepsilon_2, \ldots)$ the set of zeros of

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots$$

inside $\{z : |z| < 1 - \delta\}$. To make a continuous map of this requires some manipulation.

By Jensen's theorem, as was shown in Section 2, there is an upper bound n on the number of zeros that a power series with 0, 1 coefficients can have inside $\{z : |z| < 1 - \delta\}$. Let M be $\{z : |z| \leqslant 1\}$ with the annulus

$\{z : 1 - \delta \leqslant |z| \leqslant 1\}$ shrunk to a point P. (Therefore M is topologically a sphere.) To each power series $1 + \sum_{i=1}^{\infty} \varepsilon_i z^i$, $\varepsilon_i \in \{0, 1\}$, we assign the set of zeros inside $\{z : |z| < 1 - \delta\}$, (counted with multiplicities) and throw in extra copies of the point P as necessary to bring the total number of points to n. Since the order of these n elements of M is unspecified, we obtain a point of M^n/S_n. Let $f((\varepsilon_1, \varepsilon_2, \ldots))$ be this point.

We claim that this map

$$f : \{0, 1\}^{\omega} \to M^n/S_n$$

is continuous. This follows easily from Rouché's theorem; if two power series agree in the first m coordinates for m sufficiently large then their zeros inside $\{z : |z| < 1 - \delta\}$ will be within ε. Some may escape or enter the disk, but this is not a problem, since in the topology on M, P is close to all points z with $|z|$ sufficiently near $1 - \delta$.

We next check condition (4.1) of Lemma 4.1. This is easily done using the following trick: given

$$v = (v_1, v_2, \ldots, v_n) \in \{0, 1\}^n,$$

let $w = (v_1, v_2, \ldots, v_n, 1, v_1, v_2, \ldots, v_n)$. Then $v \in S_{v0}$, $w \in S_{v1}$, and $f(v) = f(w)$ (we extend v, w to infinite vectors by appending 0's), since

$$1 + v_1 z + v_2 z^2 + \cdots + v_n z^n$$

and

$$\begin{aligned} 1 + v_1 z + v_2 z^2 + \cdots + v_n z^n + z^{n+1} + v_1 z^{n+2} + \cdots + v_n z^{2n+1} \\ = (1 + v_1 z + v_2 z^2 + \cdots + v_n z^n)(1 + z^{n+1}) \end{aligned}$$

have the same zeros inside $\{z : |z| < 1 - \delta\}$. Therefore we may apply Lemma 4.1 and deduce that the image of f is path connected.

Since $f((0, 0, \ldots)) = (P, P, P, \ldots, P)$, we may apply Lemma 4.2 with $A = \text{image}(f)$ to deduce that $\overline{W}_\delta$ with the annulus $\{z : 1 - \delta \leqslant |z| \leqslant 1\}$ shrunk to a point P is a connected subset of M. This is equivalent to the connectivity of $\overline{W}_\delta$.

Since $\overline{W} \cap \{z : |z| \leqslant 1\}$ is the decreasing intersection of the compact connected sets $\overline{W}_{1/m}$, it too is connected. So is its image under $z \mapsto 1/z$. Finally, $\overline{W}$ is the union of these two sets, which meet on the unit circle, so $\overline{W}$ is connected as well. $\square$

5. $\overline{W}$ IS PATH CONNECTED

Here we refine the argument of the previous section to prove $\overline{W}$ is path connected. There are two main difficulties that arise. One is that the path connected analogue of Lemma 4.2, although still true (at least when M is Hausdorff), is much harder to prove. The second is that a decreasing intersection of compact path connected sets need not be path connected, so we can no longer restrict our attention to the zeros within $\{z : |z| < 1 - \delta\}$.

The lifting lemma below will be used as a substitute for Lemma 4.2. Its proof is based on proofs obtained independently by David desJardins and Emanuel Knill.

LEMMA 5.1. (Lifting lemma): *Let M be a Hausdorff space and let $\pi : M^n \to M^n/S_n$ be the projection map. Let $f : [0, 1] \to M^n/S_n$ be a continuous map. Then there is a continuous map $g : [0, 1] \to M^n$ such that $f = \pi \circ g$.*

SUBLEMMA 5.1. *Let $\Delta = \{t \in [0, 1] : f(t)$ consists of n copies of a single point$\}$. Let $g : [0, 1] \to M^n$ be an arbitrary function that is a lift of f. Then g is automatically continuous at all $t_0 \in \Delta$.*

Proof. Suppose $t_0 \in \Delta$ and $f(t_0) = \{x, x, \ldots, x\}$. If U is an open neighborhood of x,

$$g^{-1}(U^n) = f^{-1}(\pi(U^n))$$

which is open. Since such subsets U^n form a neighborhood base at $(x, x, \ldots, x) \in M^n$, this proves that g is continuous at t_0.

SUBLEMMA 5.2. *Let I_1, I_2 be closed subintervals of $[0, 1]$ such that $I_1 \cap I_2$ is a single point $\{t\}$. If g_j is a continuous lift of f on $I_j (j = 1, 2)$ then there is a continuous lift g of f on $I_1 \cup I_2$ such that $g|_{I_1} = g_1$.*

Proof. Since $g_1(t)$ and $g_2(t)$ differ only by a permutation, we can compose g_2 with a permutation $\sigma : M^n \to M^n$ and then paste the result to g_1.

SUBLEMMA 5.3. *The conclusions of Sublemma 5.2 hold even if I_1 and I_2 intersect in more than a point.*

Proof. This follows form Sublemma 5.2 since $I_1 \cup I_2$ can be expressed as the union of I_1 with at most two closed subintervals of I_2 each meeting I_1 in a point.

SUBLEMMA 5.4. *If I is a closed subinterval of $[0, 1]$ and every $t \in I$ has a neighborhood on which f has a lift, then f has a lift on I.*

Proof. By compactness, we can cover I by closed intervals $I_1, I_2, \ldots, I_k$ on which f has a lift, and we may assume $I_j \cap I_{j+1} \neq \varnothing$ for $1 \leqslant j \leqslant k$. By induction on j, Sublemma 5.3 lets us extend the lift on I_1 to a lift on $I_1 \cup I_2 \cup \cdots \cup I_j$.

SUBLEMMA 5.5. *The same holds if I is any subinterval of $[0, 1]$.*

Proof. Let $C_1 \subseteq C_2 \subseteq \cdots$ be closed intervals such that $\bigcup_{i=1}^{\infty} C_i = I$. By Sublemma 5.4, there is a lift on each C_i. By repeated use of Sublemma 5.3, extend the lift on C_1 to a lift on C_2, extend this to C_3, etc. This process gives a lift on I.

Proof of Lemma 5.1. We use induction on n. The case $n = 1$ is trivial, so assume $n > 1$. By Sublemma 5.1, it suffices to find a lift on each connected component I of $[0, 1] \backslash \Delta$. By Sublemma 5.5 it suffices to show that any $t_0 \in I$ has a neighborhood on which there is a lift.

Suppose $z_1, z_2, \ldots, z_k (k \geqslant 2)$ are the distinct elements of the multiset $f(t_0)$, occurring with multiplicities $n_1, n_2, \ldots, n_k$ respectively. Since M is Hausdorff, there exist pairwise disjoint neighborhoods U_i of z_i. Let N be a closed interval neighborhood of t_0 such that $t \in N$ implies $f(t) \in \pi(U_1^{n_1} \times \cdots \times U_k^{n_k})$. Then on N, we can lift f to a path $\tilde{f}$ in $M^{n_1}/S_{n_1} \times \cdots \times M^{n_k}/S_{n_k}$ since the projection

$$M^{n_1}/S_{n_1} \times \cdots \times M^{n_k}/S_{n_k} \to M^n/S_n$$

restricts to a homeomorphism on the projections of $U_1^{n_1} \times \cdots \times U_k^{n_k}$. By the inductive hypothesis applied to each of the k coordinates of $\tilde{f}$, we can lift $\tilde{f}$ to a path in $M^{n_1} \times \cdots \times M^{n_k} = M^n$ as desired. $\square$

THEOREM 5.1. *$\bar{W}$ is path connected.*

Proof. Let M be $\{z : |z| \leqslant 1\}$ with the unit circle shrunk to a point P. Again M is topologically a sphere, so we may give it a bounded metric d.

Let M^∞ be the set of sequences $x = \{x_i\}_{i=1}^\infty$ which converge to P and define a metric d_∞ on M_∞ by

$$d_\infty(x, y) = \sup_i d(x_i, y_i) .$$

Let the group S_∞ of permutations of $\{1, 2, \ldots,\}$ act on M^∞ by permuting the coordinates. Define a metric D on the quotient space M^∞/S_∞ by letting

$$D(\bar{x}, \bar{y}) = \inf_{\sigma \in S_\infty} d_\infty(x, \sigma y) .$$

Here $(\bar{x}, \bar{y})$ denote the projections of $x, y \in M^\infty$ to M^∞/S_∞. (That $D(\bar{x}, \bar{y}) = 0$ if and only if $\bar{x} = \bar{y}$ requires the convergence of x, y.) The set of zeros of a power series

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots$$

inside $\{z : |z| < 1\}$ forms a sequence in M converging to P (by Proposition 2.1) or else is finite, in which case we append an infinite sequence of P's. This defines a map

$$f : \{0, 1\}^\omega \to M^\infty/S_\infty .$$

By the same Rouché's theorem argument used in the proof of Theorem 4.1, this map is continuous. The conditions of Lemma 4.1 hold for the same reason as before, so the image of f is path connected.

Suppose $z_0 \in \overline{W} \cap \{z : |z| < 1\}$. Let $\omega : [0, 1] \to M^\infty/S_\infty$ be a path from the image under f of a 0, 1 power series vanishing at z_0 to $f((0, 0, \ldots)) = \{P, P, P, \ldots\}$.

Fix $m \geqslant 1$, and let M_m be $\{z : |z| \leqslant 1\}$ with the annulus

$$\{z : 1 - 1/m \leqslant |z| \leqslant 1\}$$

shrunk to a point Q. Define $|\,|$ on M_m by letting $|Q| = 1 - 1/m$. By Proposition 2.1 there is an upper bound n on the number of zeros of a 0, 1 power series inside $\{z : |z| < 1 - 1/m\}$. The path ω induces a path

$$\omega_m : [0, 1] \to (M_m)^n/S_n .$$

(Apply the projection $M \to M_m$ to each element of $\omega(t)$, and throw away infinitely many Q's to get $\omega_m(t)$.)

Pick $m_0 \geqslant 1$ such that $|z_0| < 1 - 2/m_0$. We define inductively a sequence of paths

$$\tilde{\omega}_m : [0, t_m] \to \overline{W}, \qquad m = m_0, m_0 + 1, \ldots ,$$

each extending the one before. First apply Lemma 5.1 to lift ω_{m_0} to a path $[0, 1] \to M^n_{m_0}$. Since some coordinate of $\omega_{m_0}(0)$ is z_0 and since all coordinates of $\omega_{m_0}(1)$ are Q, we get a path $\tilde{\omega}_m$ from z_0 to Q in M_{m_0}. Let t_{m_0} be the smallest $t \in [0, 1]$ such that $|\tilde{\omega}_{m_0}(t)| \geqslant 1 - 2/m_0$. Then by restriction to $[0, t_{m_0}]$ we get a path $\tilde{\omega}_{m_0}$ in $\mathbf{C}$ since $\{z \in M_{m_0} : |z| \leqslant 1 - 2/m_0\}$ can be identified with $\{z \in \mathbf{C} : |z| \leqslant 1 - 2/m_0\}$. Finally, since $\tilde{\omega}_{m_0}(t)$ is always a coordinate of $\omega(t)$, $\tilde{\omega}_{m_0}(t) \in \overline{W}$ for all $t \in [0, t_{m_0}]$.

By the same process, we inductively find for each $m > m_0$ a path $\tilde{\omega}_m : [t_{m-1}, 1] \to M_m$ such that $\tilde{\omega}_m(t_{m-1}) = \tilde{\omega}_{m-1}(t_{m-1}$. Let t_m be the smallest $t \geqslant t_{m-1}$ such that

$$|\tilde{\omega}_m(t)| \geqslant 1 - \frac{2}{m},$$

and obtain a path

$$\tilde{\omega}_m : [t_{m-1}, t_m] \to \overline{W}$$

which we append to $\tilde{\omega}_{m-1}$ to obtain

$$\tilde{\omega}_m : [0, t_m] \to \overline{W}$$

such that $\tilde{\omega}_m(t)$ is always a coordinate of $\omega(t)$.

Let $t_\infty = \sup_m t_m$. Piecing together the $\tilde{\omega}_m$'s gives a continuous map

$$\tilde{\omega} : [0, t_\infty) \to \overline{W}$$

such that $\tilde{\omega}(t)$ is a coordinate of $\omega(t)$ for all $t \in [0, t_\infty)$. The set of limit points of $|\tilde{\omega}(t)|$ as $t \to t_\infty$ is a closed interval I. Let $\omega(t_\infty) = \{z_1, z_2, z_3, \ldots\}$. If $r \in [0, 1)$ is distinct from $|z_1|, |z_2|, |z_3|, \ldots$ then $\sup_i |r - |z_i|| > 0$ (since $\{z_i\} \to P$) and by continuity of ω, r also differs by some ε from all coordinates of $\omega(t)$ for t in a neighborhood of t_∞, so r cannot be a limit point of $|\tilde{\omega}(t)|$ as $t \to t_\infty$. Thus $I \subseteq \{1, |z_1|, |z_2|, \ldots\}$ but $|z_i| \to 1$ so I must be a single point. Since $|\tilde{\omega}(t_m)| = 1 - 2/m$ for $m \geqslant m_0$, $\lim_{t \to t_\infty} |\tilde{\omega}(t)| = 1$.

Case 1: 1 is the only limit point of $\tilde{\omega}(t)$ as $t \to t_\infty$. Then $\tilde{\omega}$ extends to a path $[0, t_\infty] \to \overline{W}$ from z_0 to 1.

Case 2: There is a limit point $\theta \neq 1$, $|\theta| = 1$, of $\tilde{\omega}(t)$ as $t \to t_\infty$. By Theorem 3.1, there is an open disc centered at θ contained in $\overline{W}$. For some $t < t_\infty$, $\tilde{\omega}(t)$ is in this disc, so we can replace the tail end of $\tilde{\omega}$ on $[t, t_\infty)$ by a straight line from $\tilde{\omega}(t)$ to θ in $\overline{W}$.

In either case we can connect z_0 to a point on the unit circle via a path in $\overline{W}$. The same is true if $z_0 \in \overline{W}$, $|z_0| > 1$, since $\overline{W}$ is closed under $z \mapsto 1/z$. Since $\overline{W}$ contains the unit circle, this proves that $\overline{W}$ is path connected. $\square$

6. GRAPHS, COMPUTATIONS, AND THE SHAPE OF $\overline{W}$

The computations of zeros graphed in our figures were performed in double precision (approx. 18 decimal places) on a Silicon Graphics workstation. Some of the zeros were checked for accuracy by recomputing them in double precision (approx. 28 decimal places) on a Cray X-MP. The zero-finding program used the Jenkins-Traub algorithm and was taken from a standard subroutine library. Checks showed that the values that were obtained were accurate on average to at least 10 decimal places, which was sufficient for our graphs. The program that was used appeared to produce accurate values on the Cray for the zeros for polynomials of degrees up to about 150. (Computation of zeros of polynomials of much higher degree would have required better algorithms, cf [9].)

Zeros of a large set of random polynomials $f(z) \in P$ of degree 100 were computed on the Cray, and they exhibit most of the features visible in Figures 1-3. However, they are not as interesting as the lower degree zeros that are exhibited in Figures 1-3. The "spikes" or "tendrils" that generate the fractal appearance in the graphs we include come from a small fraction of the polynomials. Sampling even 10^4 of the 2^{99} polynomials $f(z) \in P$ of degree 100 does not yield a good representation of the extremal features that we expect to see for high as well as low degrees.

Graphs were prepared using the S system [2].

The graphs in Figures 4-6 were prepared differently. A program was written that checked whether a given w with $|w| < 1$ is in $\overline{W}$. Note that

$$\left|\sum_{k=0}^{\infty} a_k w^k\right| \leqslant B = \max(1, |1 + w|)/(1 - |w|^2)\,, \tag{6.1}$$

where the a_k are any 0, 1 coefficients, since we can write

$$\sum_{k=0}^{\infty} a_k w^k = (a_0 + a_1 w) + (a_2 + a_3 w) w^2 + \cdots .$$

The procedure was to test all sets of 0, 1 coefficients $a_1, \ldots, a_{120}$ to see whether they could be the initial segment of coefficients of a power series

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \tag{6.2}$$

for which $f(w) = 0$. Let us regard the strings of coefficients $a_1, \ldots, a_{120}$ as the leaves of a balanced binary tree, with the nodes below the root corresponding to a_1, those below to a_1, a_2, etc. The procedure was to explore this tree, checking whether

$$(6.3) \qquad \left|1 + \sum_{j=1}^{d} a_j w^j\right| > |z|^{d+1} B$$

at any stage. If (6.3) is satisfied, then w is not a zero of any power series of the form (6.2) with initial coefficients $1, a_1, \ldots, a_d$, and the subtree of that node does not have to be explored. If all the leaves are discarded by this procedure, we have a rigorous proof that $w \notin \bar{W}$, and so in fact an open neighborhood of w is outside $\bar{W}$. On the other hand, if a leaf was found with

$$(6.4) \qquad \left|1 + \sum_{j=1}^{120} a_j w^j\right| < |z|^{121} B/10 ,$$

then the program assumed that $w \in \bar{W}$. (By establishing lower bounds for the derivative of the polynomial $1 + \sum_{1}^{120} a_j z^j$ at w and using crude upper bounds for the second derivative, one could in principle prove that there is some point w' close to w such that $w' \in \bar{W}$, although the 10 in condition (6.4) might have to be decreased. Another way to prove this would be to use Lemma 3.1. This step was not carried out.) Figures 4-6 were produced by testing each w in a 1936×1936 or a 1944×1944 grid (corresponding to the resolution of our laser printer). There were few points w for which neither condition (6.3) nor condition (6.4) held. The exceptions occur primarily in Figure 4, but they do not affect how the picture looks. Had we used a tree of depth 80, the exceptions would have been much more frequent.

The computations of Figures 4-6 are not completely rigorous in that the determination of $w \notin \bar{W}$ is rigorous, while that of $w \in \bar{W}$ is not. Moreover, an implicit premise in the preparation of Figures 4-6 was that if a point $w \in \bar{W}$, then the whole neighborhood of w represented by the corresponding pixel is in $\bar{W}$. On the other hand, the computations of Figures 1-3 are rigorous.

It is possible to use computations to obtain rigorous estimates for $\bar{W}$ that are sharper than those of Theorem 2.1. As an example, we sketch how a moderate amount of straightforward computing establishes that there are no $w \in \bar{W} \setminus \mathbf{R}$ with $|w| < 0.7$. We modify the method of proof of Theorem 2.1. Write

$$(6.5) \qquad f(z) = 1 + \sum_{j=1}^{10} a_j z^j + \frac{1}{2}(z^{11} + z^{12} + \cdots) + g(z) ,$$

where

$$g(z) = \frac{1}{2} \sum_{k=11}^{\infty} \varepsilon_k z^k , \qquad \varepsilon_k = \pm 1 . \tag{6.6}$$

Then we can write

$$f(z) = F(z) + g(z) ,$$

$$F(z) = G(z)/2(1 - z) ,$$

$$G(z) = 2(1 - z)\left(1 + \sum_{j=1}^{10} a_j z^j\right) + z^{11} .$$

If we establish that $|F(z)| > |g(z)|$ on some simple closed contour about the origin, then by Rouché's theorem $f(z)$ and $F(z)$ will have the same number of zeros inside that contour. To prove that $|F(z)| > |g(z)|$ on a contour C, it suffices to show that $|F(z)| > g(z) + \delta$ on a discrete set of points z on C, where $\delta > 0$ is such that bounds on the derivatives of $F(z)$ and $g(z)$ guarantee that $|F(z)| - |g(z)|$ will not decrease by more than δ between the sampling points. This was applied to each of the 2^{10} choices of $a_1, \ldots, a_{10}$. Of the 1024 functions $F(z)$, 997 satisfied $|F(z)| > |g(z)|$ on

$$C_3 = \{z : |z| = 0.7\} .$$

The remaining 27 functions $F(z)$ were shown to satisfy $|F(z)| > |g(z)|$ on the contour

$$\begin{aligned} C_4 = \{z : |z| = 0.7, \quad |y| \geqslant 0.04\} \\ \cup \{z : x = -0.74, \ |y| \leqslant 0.04\} \\ \cup \{z : |y| = 0.04, \ -0.74 \leqslant x \leqslant -0.6, \ |z| \geqslant 0.7\} . \end{aligned}$$

Finally, zeros of each of the 1024 polynomials $G(z)$ were computed, and it was found that 85 of these polynomials had a single zero in $|z| \leqslant 0.74$, and the remaining 939 had none. Thus in all cases we can conclude that $f(z)$ has at most one zero in $|z| \leqslant 0.7$. Such a zero has to be real.

The estimates used above were crude, and with more care one can either decrease the amount of computing (and even eliminate it altogether) or obtain better bounds for $\bar{W}$.

The basic principle that makes it possible to obtain good estimates of $\bar{W}$ is that for extremal points $w \in \bar{W}$, the power series $f(z)$ with 0, 1 coefficients such that $f(w) = 0$ are restricted. For example the region depicted in Figures 3 and 4 is

$$V = \{z = x + iy : -0.501 \leqslant x \leqslant -0.497, 0.537 \leqslant y \leqslant 0.541\} .$$

Numerical computation (evaluating polynomials of degrees $\leqslant 9$ with 0, 1 coefficients at a 41×41 uniform grid, and bounding derivatives) shows that if $w \in V \cap \overline{W}$, then w can only be a zero of a power series of the form

$$f(z) = 1 + z + z^2 + z^4 + z^7 + z^9 + \sum_{k=10}^{\infty} a_k z^{10} .$$

This restricted the set of $f(z)$ that had to be considered, and made possible the computation of Figure 3, as it would not have been feasible to examine all polynomials of degrees $\leqslant 32$. Furthermore, this restriction on the coefficients of $f(z)$ makes it possible to estimate the shape of $V \cap \overline{W}$.

It should be possible to prove rigorously, with the help of numerical computations, such as those mentioned above, that the hole in $\overline{W}$ mentioned in the Introduction and pictured in Figure 6 is isolated in the sense that there is a continuous closed curve in $\overline{W} \cap U$, for U a small rectangle, that encloses the hole. We have not done this.

To explain the fractal appearance of $\overline{W}$, suppose that $w \in W$, $|w| < 1$, and that $f(w) = 0$ where

$$f(z) = 1 + \sum_{j=1}^{d} a_j z^j , \qquad a_j = 0, 1 .$$

Suppose that

$$g(z) = f(z) + z^{d+1} \sum_{k=0}^{\infty} b_k z^k , \qquad b_k = 0, 1 .$$

If $g(z) = 0$ and $|z - w|$ is small, while d is large, we have

$$\begin{aligned} 0 = g(z) &\cong g(w) + (z - w) g'(w) \\ &\cong w^{d+1} \sum_{k=0}^{\infty} b_k w^k + (z - w) f'(w) . \end{aligned}$$

If $f'(w) \neq 0$ (which as far as we know may hold for all w with $|w| < 1$), then $g'(w) \neq 0$ for d large enough, and we can expect that

$$z \cong w - \frac{w^{d+1} \sum_{k=0}^{\infty} b_k w^k}{f'(w)} .$$

Thus if

$$Q(w) = \left\{ \sum_{k=0}^{\infty} b_k w^k : b_k = 0, 1 \right\} ,$$

then we expect to find zeros in a neighborhood of each point of

$$w - w^{d+1}(f'(w))^{-1}Q(w) \, .$$

The set $Q(w)$ is connected [1], and for $w \notin \mathbf{R}$, it seems that it contains a small disk around the origin. The set $Q(w)$ is a continuous function of w, which accounts for the similarity of the protrusions from $\overline{W}$ visible in Figures 5 and 6. (The protrusions in Figure 4 are different, since there the sets $Q(w)$ are of different shape from those in Figures 5 and 6.)

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