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II. THE HEART OF THE MATTER

II.0. DEFINITIONS, NOTATIONS

Given a set $E \subset \mathbf{C}^n$, not necessarily a manifold, *a wedge* $W(= W(E, \Gamma, \rho))$ *with edge* E is defined in the following way.

For Γ a nonempty open cone in $\mathbf{C}^n$ and $\rho > 0$, one sets

$$W = \{e + \gamma \in \mathbf{C}^n, e \in E, \gamma \in \Gamma, |\gamma| < \rho\} .$$

Remark. As we will see below the words wedge and edge may be confusing. Part of the edge may very well be in the interior of the wedge, we will in fact take advantage of this situation. In case of E a germ of manifold one can instead take a cone Γ in a transverse (e.g. the normal) space, if one allows shrinking the two definitions are "equivalent". And in case E is a hypersurface, a wedge contains locally at least one of the two sides of the hypersurface.

For $\varepsilon \in [0, 1)$ set

$$\mathbf{R}^2_\varepsilon = \{(s_1 + i\varepsilon s_2, s_2 - i\varepsilon s_1) \in \mathbf{C}^2, (s_1, s_2) \in \mathbf{R}^2\} .$$

This is a tilted copy of $\mathbf{R}^2$. Set

$$\Sigma_\varepsilon = \bigcup_{0 < \varepsilon' < \varepsilon} \mathbf{R}^2_{\varepsilon'} .$$

And for $R > 0$ let

$$\Sigma^R_\varepsilon = \Sigma_\varepsilon \cap B(0, R)$$

($B(0, R)$ the open ball centered at 0, and of radius R).

II.1

The following basic (and easy) fact is at the root of Trepreau's example.

LEMMA 1. *Let* W *be any wedge in* $\mathbf{C}^2$, *with edge* Σ^R_ε. *Then every point in* $\Sigma^R_\varepsilon - \{0\}$ *belongs to the interior of the polynomial hull of* $\bar{W}$.

See Proposition 2 in V for a better result. But notice that the wedge W is really needed. It is wrong that the polynomial hull of $\overline{\Sigma^R_\varepsilon}$ contains $\Sigma^R_\varepsilon - \{0\}$ in its interior. Indeed, the function $z_1^2 + z_2^2$ is real on $\Sigma_\varepsilon (z_1^2 + z_2^2 = (1 - \varepsilon^2)(s_1^2 + s_2^2))$, hence on the polynomial hull of $\overline{\Sigma^R_\varepsilon}$ (which is in fact equal to $\overline{\Sigma^R_\varepsilon}$).

Proof of Lemma 1. On Σ_ε, $z_1^2 + z_2^2 \geqslant 0$. We then foliate Σ_ε by the level sets of $z_1^2 + z_2^2$. Fix $(a, b) \in \Sigma_\varepsilon^R - \{0\}$, set $r = \sqrt{a^2 + b^2}$. Let $A = \{(z_1, z_2) \in \Sigma_\varepsilon^R, z_1^2 + z_2^2 = r^2\}$. This is an annulus in the holomorphic curve $z_1^2 + z_2^2 = r^2$, with the nonholomorphic parametrization:

$$(\varepsilon', \theta) \mapsto \frac{r}{\sqrt{1 - \varepsilon'^2}} (\cos\theta + i\varepsilon' \sin\theta, \sin\theta - i\varepsilon' \cos\theta)$$

$0 < \varepsilon' < \varepsilon_1$, $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, with $\varepsilon_1 = \min\left(\varepsilon, \sqrt{\frac{R^2 - r^2}{R^2 + r^2}}\right)$. (Such annuli appear in [11]). Write

$$(a, b) = (\sigma_1 + i\delta\sigma_2, \sigma_2 - i\delta\sigma_1), (\sigma_1, \sigma_2) \in \mathbf{R}^2 (0 < \delta < \varepsilon_1).$$

Let Y be the circle

$$\mathrm{Y} = \{(s_1 + i\delta s_2, s_2 - i\delta s_1) \in \mathbf{C}^2, (s_1, s_2) \in \mathbf{R}^2, s_1^2 + s_2^2 = \sigma_1^2 + \sigma_2^2\}.$$

The circle Y is entirely in the annulus A. Now, we make a trivial but crucial remark.

CLAIM. *There are points in* Y *which are in the wedge* W *(hence in the interior of the polynomial hull!).*

We now check the claim. The wedge W contains the wedge W_δ with edge $\mathbf{R}_\delta^2 \cap B(0, R)$ (given by the same cone Γ). Locally, after possible shrinking of R, there is no loss of generality in assuming that the cone Γ contains a non-zero vector $(it_1, it_2) \in i\mathbf{R}^2$. Normalize (it_1, it_2) so that $t_1^2 + t_2^2 = \sigma_1^2 + \sigma_2^2$. The point $(-t_2 + i\delta t_1, t_1 + i\delta t_2)$ is a point on the circle Y which is in the wedge W since it belongs to the ray with direction (it_1, it_2) and with origin at the point $(-t_2 + i\delta' t_1, t_1 + i\delta' t_2)$. Take $0 < \delta' < \delta$, $\delta - \delta'$ small enough.

The claim is proved, and Lemma 1 follows now by propagation along the holomorphic curve A. I wish to insist that here we are now dealing with one of the most primitive versions of propagation along holomorphic curve. It is only with the goal of having a paper elementary and as self contained as possible that I state and prove Lemma 2 to be applied to $M = \Sigma_\varepsilon^R - \{0\}$, to get Lemma 1.

LEMMA 2. *Let* M *be a (piece of)* C^1 *hypersurface in* $\mathbf{C}^2$. *Let* W *be a wedge with edge* M *(i.e. at each point of* M, W *contains at least one side of* M). *Let* C *be a holomorphic curve in* M. *If some point* *of* C *belongs to the interior of the polynomial hull of* $\bar{W}$, *then* C *is entirely included in the interior of the polynomial hull of* $\bar{W}$.

By holomorphic curve we will mean a connected 1-dimensional holomorphic manifold.

Proof. Let O be the interior of the polynomial hull of $\bar{W}$. It has to be shown that the set of points $p \in C$ which belong to O is closed in C. It is obviously open. Things being so localized one has to face the following situation: a "small" analytic disk given by a holomorphic parametrization $\varphi: \bar{\Delta} \to C$ (Δ the unit disk in $\mathbf{C}$) so that $\varphi(1) \in O$, U^+ a side of M included in W (at least one of the two sides is such) hence in O, in some neighborhood of $\varphi(\bar{\Delta})$; and one has to show that $\varphi(0) \in O$. Fix ψ a holomorphic map from $\mathbf{C}$ into $\mathbf{C}^n$ so that: $\psi(e^{i\theta}) \simeq -\vec{N}$ for θ outside some small neighborhood of 0 (mod 2π), where $\vec{N}$ is the unit outer normal to M (with respect to U^+), at say the point $\varphi(0)$, and $\psi(0)$ is arbitrarily chosen.

For $\eta > 0$, η small enough $\varphi(e^{i\theta}) + \eta\psi(e^{i\theta}) \in O$ for all θ, hence $\varphi(0) + \eta\psi(0) \in O$. Taking into account some uniformity with respect to $\psi(0)$, this gives Lemma 2.

III. Lifting to $\mathbf{C}^3$

We are simply going to consider sets K in $\mathbf{C}^3$ rotationally invariant in the first variable, that we describe as follows. For each $t \in [0, t_0]$ we are given a compact set $K_t \subset \mathbf{C}^2$. We consider the set $K \subset \mathbf{C}^3$ which is the closure of the set $\{(w, z_1, z_2) \in \mathbf{C}^3; (z_1, z_2) \in K_{|w|}, |w| \leqslant t_0\}$. i.e.

$$K = \overline{\bigcup_{|w| \leqslant t_0} \{w\} \times K_{|w|}}.$$

$\hat{K}$ denotes the polynomial hull of K in $\mathbf{C}^3$, while $\widehat{\cup K_t}$ denotes the polynomial hull in $\mathbf{C}^2$ of the closure of the set $\bigcup_{t \leqslant t_0} K_t$.

LEMMA 3. *Let* $(0, \zeta_1, \zeta_2) \in \mathbf{C}^3$, *the following are equivalent:*

$$\begin{cases} (i) & (0, \zeta_1, \zeta_2) \in \hat{K} \\ (ii) & (\zeta_1, \zeta_2) \in \widehat{\cup K_t}\,. \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) is trivial. We are interested in (ii) $\Rightarrow$ (i). Let $P(w, z_1, z_2)$ be a polynomial in 3 variables. To P we associate the polynomial $\tilde{P}$ defined by

$$\tilde{P}(w, z_1, z_2) = P(0, z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta} w, z_1, z_2) d\theta\,.$$