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Artikel: REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS

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Kapitel: §6. Quadratic surds

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PROBLEM 5. *Let* T *be a given Möbius map and let* $I(T)$ *be the associated interval. Let* $\zeta \in I(T)$. *To compute the Hausdorff dimension of those* x *for which*

$$\lim_{n \to \infty} \frac{\delta(Tx_n)}{n} = \zeta .$$

Extend this problem to higher dimensions in the spirit of problem 4.

§6. QUADRATIC SURDS

Let x be a real quadratic number. Its continued fraction expansion is ultimately periodic. Let $\pi(x)$ be its period. H. Cohen [3], followed by J. Cusick [4] and Paysant-Leroux [11] studied the action of a Möbius map on the period. They established that

$$\lim_{\pi(x) \to \infty} \sup \frac{\pi(Tx)}{\pi(x)} = R(\Delta)$$

where $R(\Delta)$ is an integer. Furthermore

$$A\, n \ln n \leqslant R(n) \leqslant B\, n \ln n + 1$$

for some constants $A > 0$, $B > 0$. A simple argument then shows that

$$\lim_{\pi(x) \to \infty} \inf \frac{\pi(Tx)}{\pi(x)} = \frac{1}{R(\Delta)} .$$

PROBLEM 6. *Is it true that for all real quadratic irrational* x

$$\sup_n \pi(x^n) = \infty \,?$$

Define the interval

$$J(\Delta) = \left[\frac{1}{R(\Delta)}, R(\Delta)\right] .$$

PROBLEM 7. *Let* $\zeta \in J(\Delta)$. *Prove the existence of a sequence of real quadratic numbers* x_n *with strictly increasing period such that*

$$\lim_{n \to \infty} \frac{\pi(Tx_n)}{\pi(x_n)} = \zeta .$$

Extend this result to higher dimensions as in Problem 4.

PROBLEM 8. *Does there exist a sequence* x_n *of quadratic numbers with strictly increasing period such that for all Möbius map* T

$$\lim_{n \to \infty} \frac{\pi(Tx_n)}{\pi(x_n)} = 1 \ ?$$

We believe some of our problems are relatively easy to solve. But quite obviously Problem 1, 2 and maybe 6 are deep.

REFERENCES

[1] BILLINGSLEY, P. *Ergodic Theory and Information.* Wiley 1965.

[2] CHOQUET, G. Répartition des nombres $k(3/2)^n$; et ensembles associés, (English summary) *C.R. Acad. Sci. Paris Sér. A-B 290* (1980), no. 13, A575-A580; Algorithmes adaptés aux suites $(k\theta^n)$ et aux chaines associées, (English summary) *C.R. Acad. Sci. Paris Sér. A-B 290* (1980), no. 16, A719-A724; θ-jeux récursifs et application aux suites $(k\theta^n)$; solenoïdes de T^z, (English summary) *C.R. Acad. Sci. Paris Sér. A-B 290* (1980), no. 19, A863-A868; Construction effective de suites $(k(3/2)^n)$. Etude des mesures (3/2)-stables, (English summary) *C.R. Acad. Sci. Paris Sér. A-B 291* (1980), no. 2, A69-A74; Les fermés (3/2)-stables de T; structure des fermés dénombrables; applications arithmétiques, (English summary) *C.R. Acad. Sci. Paris Sér. A-B 291* (1980), no. 4, A-239-A244; θ-fermés; θ-chaines et θ-cycles (pour $\theta = 3/2$), (English summary) *C.R. Acad. Sci. Paris Sér. I Math. 292* (1981), no. 1, 5-10; θ-fermés et dimension de Hausdorff. Conjectures de travail. Arithmétique des θ-cycles (où $\theta = 3/2$), (English summary) *C.R. Acad. Sci. Paris Sér. I Math. 292* (1981), no. 6, 339-344.

[3] COHEN, H. Multiplication par un entier d'une fraction continue périodique. *Acta Arith. 26* (1974), 129-148.

[4] CUSICK, T. Integer multiples of periodic continued fractions. *Pacific J. Math. 78* (1978), 47-60.

[5] DAUDÉ, H. Thèse, Caen 20 Janv. 1993.

[6] DIXON, J.D. The number of steps in the Euclidean algorithm. *J. of Number Theory 2* (1970), 414-422. A simple estimate for the number of steps in the Euclidean algorithm. *Amer. Math. Monthly 78* (1971), 374-376.

[7] HEILBRONN, H. On the average length of a class of finite continued fractions. In *Abhandlungen aus Zahlentheorie und Analysis,* Berlin 1968, edited by P. Turàn, 87-96.

[8] LAMÉ, G. Note sur la limite... *Comptes Rendus Acad. Sci. Paris 19* (1844), 867-870.

[9] LÉVY, P. Sur le développement en fraction continue d'un nombre choisi au hasard. *Compositio Mathem. 3* (1936), 286-303.

[10] MENDÈS FRANCE, M. Quelques problèmes relatifs à la théorie des fractions continues limitées. *Séminaire de Théorie des Nombres Bordeaux 1971-72,* 4 bis 01-09; Sur les fractions continues limitées, *Acta Arith. 23* (1973), 207-215; The depth of a rational number, in *Topics in Number Theory, Debrecen 1974, Coll. Mathem. Soc. Jànoi Bolyai 13,* 183-194.