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since it is b , the smallest of the two integers a and b that seems to control the behavior of $(a/b)^n$. It is also to be noticed that in Dixon's inequality, the $\ln a$ that appears on the left hand side could well be replaced by $\ln b$ since for "almost all" couples a, b , $\ln a \approx \ln b$.

Numerical evidence supports equality (1). Based on computer computation, Chr. Batut and M. Olivier showed that

$$\left| \frac{1}{n} \delta \left(\left(\frac{a}{b} \right)^n \right) - \frac{12}{\pi^2} \ln 2 \ln b \right|$$

is less than .02 for n in the range (4000, 5000) and for $a = 3, b = 2$ on the one hand and $a = 5, b = 2$ on the other hand.

§ 4. RELATED PROBLEMS

My initial (unsuccessful) attempts to prove Theorem 1 were based on the comparison of $\delta(ax/b)$ to $\delta(x)$. I was hoping that a relationship between both depths would give by induction some results on $\delta(xa^n/b^n)$. This turned out nonconclusive, yet I did obtain some results which I believe are interesting in themselves [10].

Let a, b, c, d be coprime integers and let $\Delta = |ad - bc|$. Consider the Möbius map

$$x \mapsto Tx = \frac{ax + b}{cx + d}.$$

THEOREM 2.

$$\limsup_{\delta(x) \rightarrow \infty} \frac{\delta(Tx)}{\delta(x)} = \Theta(\Delta)$$

where Θ takes odd integral values. As n increases to infinity $\Theta(n)$ behaver like $\ln n$. More precisely let

$$\alpha = \left(2 \ln \frac{1 + \sqrt{5}}{2} \right)^{-1}.$$

Then for all integer $n \geq 1$

$$1 + \alpha \ln n \leq \Theta(n) \leq 2(1 + \alpha \ln n).$$

Θ is linked to the depth by the formula

$$\Theta(n) = \max_{1 \leq b \leq n} \delta \left(\frac{b}{n} \right) + 1.$$

Finally

$$\liminf_{\delta(x) \rightarrow \infty} \frac{\delta(Tx)}{\delta(x)} = \frac{1}{\Theta(\Delta)}.$$

The following table gives the first values of Θ

n	1	2	3	4	5	6	7	8	9	10	...
$\Theta(n)$	1	3	3	3	5	3	5	5	5	5	...

Actually Theorem 1 can be improved. There exist two constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that for all rational x

$$\frac{1}{\Theta(\Delta)} \delta(x) - C_1 \leq \delta(Tx) \leq \Theta(\Delta) \delta(x) + C_2.$$

Both inequalities are sharp apart from the exact values of C_1 and C_2 .

5. MORE QUESTIONS

To every Möbius map T we associate the interval $I(T) = [\Theta^{-1}(\Delta), \Theta(\Delta)]$.

PROBLEM 3. *Is it true that for all $\zeta \in I(T)$ there exists a sequence of rational numbers x_n such that $\lim_{n \rightarrow \infty} \delta(x_n) = \infty$ and*

$$\lim_{n \rightarrow \infty} \frac{\delta(Tx_n)}{\delta(x_n)} = \zeta?$$

PROBLEM 4. *Let T_1, T_2, \dots, T_k be Möbius maps with pairwise coprime determinants $\Delta_1, \Delta_2, \dots, \Delta_k$.*

Is it true that for all

$$(\zeta_1, \zeta_2, \dots, \zeta_k) \in \prod_{i=1}^k I(T_i)$$

there exists a sequence of rational x_n with strictly increasing depths such that for all $i = 1, 2, \dots, k$

$$\lim_{n \rightarrow \infty} \frac{\delta(T_i x_n)}{\delta(x_n)} = \zeta_i?$$

Can k be infinite?

The following result should be mentioned at this point.