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Artikel: REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS
Autor: Mendès France, Michel
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Now let $x > 1$ be a rational number which is not an integer. Suppose

$$\sup_n \delta(x^n) < \infty$$

Then for some k

$$E(x) \subset A_{2k}$$

hence

$$E^{(k+1)}(x) = \varnothing .$$

Assuming a positive answer to Problem 1, we conclude that x is a Pisot number, i.e. a rational integer. This contradicts the assumption hence

$$\sup_n \delta(x^n) = \infty . \qquad QED$$

§3. More questions on $\delta(x^n)$

H. Heilbronn [7], T. Tonkov [15] and finally J.W. Porter [13] improving on one another established that as a tends to infinity

$$\frac{1}{\varphi(a)} \sum_{\substack{b<a \\ (a,b)=1}} \delta\left(\frac{a}{b}\right) = \frac{12}{\pi^2} \ln 2 \ln a + O(1) .$$

Independtly, J.D. Dixon [6] showed that for all $\varepsilon > 0$ and for all $a, b, 1 < b < a < x$ with the exception of at most $o(x^2)$ couples, one has

$$\left| \delta\left(\frac{a}{b}\right) - \frac{12}{\pi^2} \ln 2 \ln a \right| \leqslant (\ln a)^{\frac{1}{2}+\varepsilon} .$$

See H. Daudé's work for a dual result [5]. These results suggest the second problem.

Problem 2. *Is it true that for all coprime* a *and* $b, 1 < b < a$

$$\lim_{n \to \infty} \frac{1}{n} \delta\left(\left(\frac{a}{b}\right)^n\right) = \frac{12}{\pi^2} \ln 2 \ln b \ ? \tag{1}$$

The limit should indeed be what is stated above and not

$$\frac{12}{\pi^2} \ln 2 \ln a .$$

since it is b, the smallest of the two integers a and b that seems to control the behavior of $(a/b)^n$. It is also to be noticed that in Dixon's inequality, the $\ln a$ that appears on the left hand side could well be replaced by $\ln b$ since for "almost all" couples $a, b, \ln a \approx \ln b$.

Numerical evidence supports equality (1). Based on computer computation, Chr. Batut and M. Olivier showed that

$$\left| \frac{1}{n} \delta\left(\left(\frac{a}{b}\right)^n\right) - \frac{12}{\pi^2} \ln 2 \ln b \right|$$

is less than .02 for n in the range (4000, 5000) and for $a = 3$, $b = 2$ on the one hand and $a = 5$, $b = 2$ on the other hand.

§4. Related problems

My initial (unsuccessful) attempts to prove Theorem 1 were based on the comparison of $\delta(ax/b)$ to $\delta(x)$. I was hoping that a relationship between both depths would give by induction some results on $\delta(xa^n/b^n)$. This turned out nonconclusive, yet I did obtain some results which I believe are interesting in themselves [10].

Let a, b, c, d be coprime integers and let $\Delta = |ad - bc|$. Consider the Möbius map

$$x \mapsto Tx = \frac{ax + b}{cx + d} .$$

THEOREM 2.

$$\limsup_{\delta(x) \to \infty} \frac{\delta(Tx)}{\delta(x)} = \Theta(\Delta)$$

where Θ *takes odd integral values. As* n *increases to infinity* $\Theta(n)$ *behaver like* $\ln n$*. More precisely let*

$$\alpha = \left(2 \ln \frac{1 + \sqrt{5}}{2}\right)^{-1} .$$

Then for all integer $n \geqslant 1$

$$1 + \alpha \ln n \leqslant \Theta(n) \leqslant 2(1 + \alpha \ln n) .$$

Θ *is linked to the depth by the formula*

$$\Theta(n) = \max_{1 \leqslant b \leqslant n} \delta\left(\frac{b}{n}\right) + 1 .$$