Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 39 (1993)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS
Autor: Mendès France, Michel
Kapitel: §2. A QUESTION CONCERNING PISOT NUMBERS
DOI: https://doi.org/10.5169/seals-60426

This observation led me to ask in 1971 [10] whether it was indeed true that

$$\sup_n \delta((a/b)^n) = \infty$$

whithout any other assumptions than $1 < b < a$, $(a, b) = 1$. The problem was solved by Y. Pourchet in an unpublished letter he sent me [14] and by G. Choquet in a series of Comptes Rendus à l'Académie des Sciences [2].

THEOREM 1. *If* a *and* b *are two coprime integers* $1 < b < a$ *then*

$$\lim_{n \to \infty} \delta((a/b)^n) = \infty .$$

Choquet's proof involves dynamical systems. He could only show that the "sup" is infinite. Pourchet's proof is number theoretical and uses the Mahler-Ridout theorem which strengthens Roth's famous result on the rational approximations of algebraic numbers.

It is a pity that Pourchet never published his result. Fortunately A. van der Poorten gave some details of the proof in [16].

§2. A QUESTION CONCERNING PISOT NUMBERS

Let $x > 1$ be a real number. Define the set

$$E(x) = \{x^n \pmod 1 \mid n \in \mathbf{N}\} \subset [0, 1]$$

Let $E'(x)$ be the derived set i.e. the set of cluster points of $E(x)$. Define $E^{(n)}(x)$ recursively to be the derived set of $E^{(n-1)}(x)$, $n \geqslant 1$. In [12] Pisot establishes that if x is a real algebraic number larger that 1 such that $E''(x) = \varnothing$ then x is a Pisot number. I ask the following question.

PROBLEM 1. *Is it true that if* $x > 1$ *is algebraic and if for some* $k \in \mathbf{N}$ $(k \geqslant 2)$ $E^{(k)}(x) = \varnothing$ *then* x *is a Pisot number?*

A positive answer to this problem implies the weak form of Theorem 1, namely that the sup is infinite. Indeed, define $A_0 = \{0\}$ and for $k \geqslant 1$

$$A_{2k} = \{\zeta \in (0, 1) \mid \delta(\zeta) \leqslant 2k\} .$$

Let $A'_{2k}, A''_{2k}, \cdots$ be the derived sets of A_{2k}. Clearly $A'_{2k} = A_{2k-2}$, therefore

$$A_{2k}^{(k+1)} = \varnothing .$$

Now let $x > 1$ be a rational number which is not an integer. Suppose

$$\sup_n \delta(x^n) < \infty$$

Then for some k

$$E(x) \subset A_{2k}$$

hence

$$E^{(k+1)}(x) = \varnothing .$$

Assuming a positive answer to Problem 1, we conclude that x is a Pisot number, i.e. a rational integer. This contradicts the assumption hence

$$\sup_n \delta(x^n) = \infty . \qquad QED$$

§3. More questions on $\delta(x^n)$

H. Heilbronn [7], T. Tonkov [15] and finally J.W. Porter [13] improving on one another established that as a tends to infinity

$$\frac{1}{\varphi(a)} \sum_{\substack{b < a \\ (a,b)=1}} \delta\left(\frac{a}{b}\right) = \frac{12}{\pi^2} \ln 2 \ln a + O(1) .$$

Independtly, J.D. Dixon [6] showed that for all $\varepsilon > 0$ and for all a, b, $1 < b < a < x$ with the exception of at most $o(x^2)$ couples, one has

$$\left| \delta\left(\frac{a}{b}\right) - \frac{12}{\pi^2} \ln 2 \ln a \right| \leqslant (\ln a)^{\frac{1}{2} + \varepsilon} .$$

See H. Daudé's work for a dual result [5]. These results suggest the second problem.

Problem 2. *Is it true that for all coprime* a *and* b, $1 < b < a$

$$\lim_{n \to \infty} \frac{1}{n} \delta\left(\left(\frac{a}{b}\right)^n\right) = \frac{12}{\pi^2} \ln 2 \ln b \ ? \qquad (1)$$

The limit should indeed be what is stated above and not

$$\frac{12}{\pi^2} \ln 2 \ln a .$$