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REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS

by Michel MENDÈS FRANCE

SUMMARY. We present eight problems related to the length of continued fractions of rational numbers and to the length of the period of quadratic surds.

§1. A FRUSTRATING QUESTION

Let a and b be two coprime integers, $1 < b < a$. *Is it true that the sequence* $(a/b)^n, n = 0, 1, 2, \cdots$ *is dense* (mod 1)? This very old problem of Pisot and Vijayaraghavan is still unanswered. Pisot, Vijayaraghavan and André Weil did however show that there exist infinitely many cluster points.

Are any one of these cluster points irrational? Even this seems unanswered. We address a simpler question, but before we must define the depth of a rational number x: it is simply the length $\delta(x)$ of the continued fraction of x

$$x = [c_0, c_1, c_2, \cdots, c_\delta]$$

where we choose δ to be even ($c_\delta \geqslant 1$). For example

$$\delta(k) = 0, \; k \in \mathbf{Z}; \; \delta\left(\frac{1}{2}\right) = 2; \; \delta\left(\frac{3}{5}\right) = 4 .$$

Quite obviously $\delta(a/b) = O(\ln(b))$, $1 \leqslant b < a$ (see [8]).

Suppose that the sequence $(a/b)^n$ has an irrational cluster point ζ (mod 1). Then some subsequence $(a/b)^{n_j}$ (mod 1) tends to ζ hence

$$\delta((a/b)^{n_j}) \to \infty .$$

KEY WORDS: Continued fractions, depth, Pisot numbers, Möbius transformations, quadratic surds.

This observation led me to ask in 1971 [10] whether it was indeed true that

$$\sup_n \delta((a/b)^n) = \infty$$

whithout any other assumptions than $1 < b < a$, $(a, b) = 1$. The problem was solved by Y. Pourchet in an unpublished letter he sent me [14] and by G. Choquet in a series of Comptes Rendus à l'Académie des Sciences [2].

THEOREM 1. *If a and b are two coprime integers $1 < b < a$ then*

$$\lim_{n \to \infty} \delta((a/b)^n) = \infty .$$

Choquet's proof involves dynamical systems. He could only show that the "sup" is infinite. Pourchet's proof is number theoretical and uses the Mahler-Ridout theorem which strengthens Roth's famous result on the rational approximations of algebraic numbers.

It is a pity that Pourchet never published his result. Fortunately A. van der Poorten gave some details of the proof in [16].

§2. A QUESTION CONCERNING PISOT NUMBERS

Let $x > 1$ be a real number. Define the set

$$E(x) = \{x^n \pmod 1 \mid n \in \mathbf{N}\} \subset [0, 1]$$

Let $E'(x)$ be the derived set i.e. the set of cluster points of $E(x)$. Define $E^{(n)}(x)$ recursively to be the derived set of $E^{(n-1)}(x)$, $n \geqslant 1$. In [12] Pisot establishes that if x is a real algebraic number larger that 1 such that $E''(x) = \varnothing$ then x is a Pisot number. I ask the following question.

PROBLEM 1. *Is it true that if $x > 1$ is algebraic and if for some $k \in \mathbf{N}$ $(k \geqslant 2)$ $E^{(k)}(x) = \varnothing$ then x is a Pisot number?*

A positive answer to this problem implies the weak form of Theorem 1, namely that the sup is infinite. Indeed, define $A_0 = \{0\}$ and for $k \geqslant 1$

$$A_{2k} = \{\zeta \in (0, 1) \mid \delta(\zeta) \leqslant 2k\} .$$

Let $A'_{2k}, A''_{2k}, \cdots$ be the derived sets of A_{2k}. Clearly $A'_{2k} = A_{2k-2}$, therefore

$$A_{2k}^{(k+1)} = \varnothing .$$