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REMARKS AND PROBLEMS ON FINITE AND PERIODIC CONTINUED FRACTIONS

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SUMMARY. We present eight problems related to the length of continued fractions of rational numbers and to the length of the period of quadratic surds.

§1. A FRUSTRATING QUESTION

Let a and b be two coprime integers, $1 < b < a$. *Is it true that the sequence* $(a/b)^n, n = 0, 1, 2, \cdots$ *is dense* (mod 1)? This very old problem of Pisot and Vijayaraghavan is still unanswered. Pisot, Vijayaraghavan and André Weil did however show that there exist infinitely many cluster points.

Are any one of these cluster points irrational? Even this seems unanswered. We address a simpler question, but before we must define the depth of a rational number x: it is simply the length $\delta(x)$ of the continued fraction of x

$$x = [c_0, c_1, c_2, \cdots, c_\delta]$$

where we choose δ to be even ($c_\delta \geqslant 1$). For example

$$\delta(k) = 0,\ k \in \mathbf{Z};\ \delta\left(\frac{1}{2}\right) = 2;\ \delta\left(\frac{3}{5}\right) = 4\ .$$

Quite obviously $\delta(a/b) = O(\ln(b)), 1 \leqslant b < a$ (see [8]).

Suppose that the sequence $(a/b)^n$ has an irrational cluster point ζ (mod 1). Then some subsequence $(a/b)^{n_j}$ (mod 1) tends to ζ hence

$$\delta((a/b)^{n_j}) \to \infty\ .$$

KEY WORDS: Continued fractions, depth, Pisot numbers, Möbius transformations, quadratic surds.