Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 39 (1993)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CARROUSEL MONODROMY AND LEFSCHETZ NUMBER OF SINGULARITIES

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Kapitel: 3. Zeta-function and carrousel monodromies

DOI: https://doi.org/10.5169/seals-60425

(b) a set defined as in (4) or — if case — a similar one in a carrousel disc of order 1, or

(c) a centre of a carrousel disc of order 1 inside A_i, for some $i \in P^{(1)}$, or

(d) the centre $(0, \eta)$ of the big carrousel disc.

2.6. *Definition.* Let $\mathscr{I}^{(1)}$ be a maximal set of indices $i \in P^{(1)}$ such that, if $i_1, i_2 \in \mathscr{I}^{(1)}$, then $\hat{C}_{i_1}^{(1)} \neq \hat{C}_{i_2}^{(1)}$.

For any $i \in \mathscr{I}^{(1)}$, denote by $\delta(i)$ the carrousel disc of order 1 centred at the point $c(i) := \hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. Let $a(i)$ be an arbitrarily chosen point on the boundary $\partial\delta(i)$; it is, by definition, a regular value for l_α.

Definition. Let $a \in (D_\alpha \backslash 0) \times \{\eta\}$ and let F'_a be the fibre of l_α over a. If a is fixed by the carrousel, then the monodromy h_f restricts to an action on $H^\bullet(F'_a)$, denoted by h'_a.

With these notations, we may formulate the following:

2.7. THEOREM. *If* $f \in \mathbf{m}_{X,0}$ *and* $l \in \Omega_f$, *then:*

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \sum_{i \in \mathscr{I}^{(1)}} [\Lambda(h'_{c(i)}) - \Lambda(h'_{a(i)})] \,.$$

Proof. The Lefschetz number $\Lambda(f)$ splits into a sum, following the decomposition of the set of fixed points into connected components, see 2.5. We use a suitable open covering of a set defined as in (4) and then apply the Mayer-Vietoris exact sequence. The reason of considering the set $\mathscr{I}^{(1)}$ relies on the above discution. By a straightforward computation, using also Lemma 2.1, we get our formula. □

Notice that carrousel discs of order $\geqslant 2$ do not enter in the above formula. For the computation of $\Lambda(h'_{c(i)})$, $\Lambda(h'_{a(i)})$, we refer to Remarks 3.6. There will be an example at the end.

3. ZETA-FUNCTION AND CARROUSEL MONODROMIES

3.1. Loosely speaking, each "important point" of the carrousel disc is fixed after a finite number of turns of the carrousel. We have seen that the set of fixed points after one turn determines the Lefschetz number $\Lambda(h_f)$. So the set of fixed points after k turns is the one responsible for the number $\Lambda(h_f^k)$. It may contain a finite number of circles consisting of regular values for the map l_α. Actually, these circles do not count, as shown by Lemma 2.1 (where

h_f has to be replaced by h_f^k). By examining the proof of Theorem 2.1, we get a slightly more general result:

PROPOSITION. *Let* $k \geqslant 1$. *If* $n_{i,1} \nmid k$, $\forall i \in \{1, \dots, r\}$, *then* $\Lambda(h_f^k) = \Lambda(h^k_{f|\{l=0\}})$. □

3.2. *Definition.* Let $U \subset D_\alpha \times \{\eta\}$ and let $k_U := \min\{k \mid U$ is globally fixed by the k^{th} iteration of the carrousel$\}$. Then $k_f^{k_U}$ restricts to an action on $H^\bullet(l_\alpha^{-1}(U))$, which we denote by h'_U. We call such actions *carrousel monodromies.*

3.3. The zeta-function is determined by the set of Lefschetz numbers $\Lambda(h_f^k)$, $k \geqslant 1$, as follows (see e.g. [Mi, p. 77], [A'C-2, p. 234]). If the integers $s_1, s_2, \dots$ are inductively defined by $\Lambda(h_f^k) = \Sigma_{i|k} s_i$, $k \geqslant 1$, then the zeta-function of f is given by:

$$\xi_f(t) = \prod_{i \geqslant 1} (1 - t^i)^{-s_i/i} . \tag{5}$$

On the other hand, if $\mathscr{B}^{(k)}$ denotes some small enough neighbourhood of the set of fixed points of the k^{th} power of the carrousel, then h_f^k acts on the cohomology $H^\bullet(l_\alpha^{-1}(\mathscr{B}^{(k)}))$ and, with the definition above, we get $\Lambda(h_f^k) = \Lambda(h'_{\mathscr{B}^{(k)}})$.

Let's consider the annulus A_i, as before, in the big carrousel disc. Denote by h_{A_i} the restriction of h_f to $H^\bullet(l_\alpha^{-1}(A_i))$.

If $x \in A_i$ is fixed by some power k of the carrousel, then this power has to be a multiple of $n_{i,1}$. This remark and formula (5) yield the relation:

$$[\zeta_{h_{A_i}}(t)]^{n_{i,1}} = \zeta_{h_{A_i}^{n_{i,1}}}(t^{n_{i,1}}) . \tag{6}$$

Definition. For any $i \in \{1, \dots, r\}$, denote by $\delta(i)^{(1)}$ the carrousel disc of order 1 centred at an arbitrarily chosen point of $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$, but fixed once and for all.

Let $\mathscr{L}^{(1)} := \{\delta = \delta(i)^{(1)} \mid i \in \{1, \dots, r\}, \delta(i)^{(1)}$ is not contained in any other carrousel disc of order $1\}$. For $\delta \in \mathscr{L}^{(1)}$, denote by $a(\delta)$ an arbitrarily chosen point on the boundary $\partial\bar{\delta}$.

Then we have the next recursive formula:

3.4. THEOREM. $\zeta_f(t) = \zeta_{f|\{l=0\}}(t) \cdot \prod_{\delta \in \mathscr{L}^{(1)}} \zeta_{h'_\delta}(t^{n_{i,1}}) \cdot \zeta^{-1}_{h'_{a(\delta)}}(t^{n_{i,1}})$.

Proof. We apply Mayer-Vietoris exact sequences to the covering of the carrousel disc described before. Since the fixed circles do not count for the

Lefschetz numbers, we get that the zeta-function is a product over all different annuli, each factor being of the form $\zeta_{h_{A_i}}(t)$.

We employ the notations in 2.5. Notice that the set $\mathscr{K}_i^{(1)}$ is well defined for any $i \in \{1, \ldots, r\}$. One can easily show that A_i retracts to the subset $\mathscr{R}_i := S_i \cup \bigcup_{\delta \in \mathscr{K}_i^{(1)}} \delta$, hence $\zeta_{h_{A_i}^{n_{i,1}}}(t) = \zeta_{h'_{\mathscr{R}_i}}(t)$.

If $\delta \in \mathscr{K}_i^{(1)}$, then notice that there are $n_{i,1}$ carrousel discs in A_i of the same radius as δ; if δ_1, δ_2 are any two of them, then $\zeta_{h'_{\delta_1}}(t) = \zeta_{h'_{\delta_2}}(t)$.

An open covering of $\mathscr{R}_i$ and a Mayer-Vietoris argument lead to the conclusion:

$$\zeta_{h'_{\mathscr{R}_i}}(t) = \prod_{\delta \in \mathscr{L}^{(1)}} [\zeta_{h'_\delta}(t)]^{n_{i,1}} \cdot [\zeta_{h'_{a(\delta)}}(t)]^{n_{i,1}} .$$

Using (6), our formula is now proved. Notice that the factor $\zeta_{f_{|\{l=0\}}}(t)$ corresponds to the disc A_0 defined in 1.8. $\square$

It is not hard to figure out how the process started in the proof above may continue. We apply Theorem 3.4 with h_f replaced by h'_δ and get a formula for the zeta-function $\zeta_{h'_\delta}(t)$, for any $\delta \in \mathscr{L}^{(1)}$. In a finite number of steps, going through the carrousel discs of order $1, 2, \ldots, m$, where $m := \max\{g_i \mid i \in \{1, \ldots, r\}\}$, we get a formula for $\zeta_f(t)$. To write it down, we need just the following notations.

Definition. Let $\delta(i)^{(k)}$ denote the carrousel disc of order k centred at a fixed (arbitrarily chosen) point of the set $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$. (This later set contains exactly $n_{i,1} \cdots n_{i,k}$ points). Denote $\mathscr{C}(\Delta') := \{\delta(i)^{(k)} \mid i \in \{1, \ldots, r\}, k \in \{1, \ldots, m\}\}$.

For any $\delta \in \mathscr{C}(\Delta')$, denote by $c(\delta)$ its centre and by $a(\delta)$ an arbitrarily chosen point on $\partial\bar{\delta}$.

Let $\delta \in \mathscr{C}(\Delta')$, where $\delta = \delta(i)^{(k)}$, for some indices i and k as above. Then define $n(\delta) := n_{i,1} \cdots n_{i,k}$.

Thus we get the following general zeta-function formula:

3.5. THEOREM. $\zeta_f(t) = \zeta_{f_{|\{l=0\}}}(t) \cdot \prod_{\delta \in \mathscr{C}(\Delta')} \zeta_{h'_{c(\delta)}}(t^{n(\delta)}) \cdot \zeta^{-1}_{h'_{a(\delta)}}(t^{n(\delta)})$. $\square$

By using a decreasing induction, $\zeta_f(t)$ will become finally a product of integer powers of cyclotomic polynomials.

3.6. *Remarks.* (a) The points $a(\delta), \delta \in \mathscr{C}(\Delta')$ may also be defined as follows (the precise details are left to the reader):

Let $\delta = \delta(i)^{(k)}$ and let $\hat{C}_i^{(k)}$ be (formally) defined by the equation (see (3)): $u_i = a_{k_i}\lambda^{m_{i,1}/n_{i,1}} + \cdots + \sum_{l=1}^{l_k} b_{k,l}\lambda^{(m_k+l)/n_{i,1}\cdots n_{i,k}}$. Then define a curve $G_{i,k}$, by slightly perturbing in this equation just the last coefficient b_{k,l_k}, such that $G_{i,k} \neq \hat{C}_j^{(k)}$, $\forall j \in \{1, \ldots, r\}$. For $k = g_i$, we cut the Puiseux expansion at a sufficiently high power of λ and modify the last coefficient. It follows that $a(\delta(i)^{(k)})$ may be identified to the point in $G_{i,k} \cap (D_\alpha \times \{\eta\})$ which is in the closest neighbourhood of $c(\delta(i)^{(k)})$.

(b) Let $\delta := \delta(i)^{(k)}$. Then $c(\delta)$ is a regular value for the map l_α if and only if, for any $j \in \{1, \ldots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, we have $g_j > k$. It is possible that $a(\delta)$ cannot be chosen arbitrarily close to $c(\delta)$, see also Remark 1.6.

(c) The carrousel monodromies $h'_{c(\delta)}$, $h'_{a(\delta)}$ may be defined as monodromies of functions. This remark was used by Lê in his proof of the Monodromy Theorem [Lê-1], see also [Lo]. For instance, if $\delta = \delta(i)^{(k)}$ and $(u_i^{(k)}(t), \lambda(t))$ is the parametrization of $\hat{C}_i^{(k)}$ defined in 1.5, then the pull-back diagram:

$$\begin{array}{ccc} (\mathbf{X}_i^{(k)}, 0) & \rightarrow & (\mathbf{X}, 0) \\ f_i^{(k)} \downarrow & & \downarrow \Phi \\ (\mathbf{C}, 0) & \xrightarrow{(u_i^{(k)}, \lambda)} & (\mathbf{C}^2, 0) \end{array} \tag{7}$$

defines a space $(\mathbf{X}_i^{(k)}, 0)$ and a function $f_i^{(k)}$ on it. Then $h'_{c(\delta)}$ is the monodromy of $f_i^{(k)}$.

3.7. We illustrate the formula on the following particular case: *any component Δ_i has just one Puiseux pair,* i.e. $g_i = 1$, $\forall i \in \{1, \ldots, r\}$. We assume, for the sake of simplicity, that the sets of components of $\Gamma(l, f)$ and $\Delta(l, f)$ are in one-to-one correspondence.

In this case, we have $\hat{C}_i^{(1)} = \Delta_i$ and a carrousel disc $\delta(i)^{(1)}$ is an arbitrarily small disc centred at $c(\delta(i)^{(1)}) \in \Delta_i \cap (D_\alpha \times \{\eta\})$, which is pointwise fixed by the $n_{i,1}$th iterate of the big carrousel. It follows that the point $a(\delta(i)^{(1)})$ can be chosen arbitrarily close to $c(\delta(i)^{(1)})$. The centres $c(\delta)$, $\delta \in \mathscr{C}(\Delta')$ are critical values for the map l_α. Let $c(i)$ denote a fixed, arbitrarily chosen point of the set $\Delta_i \cap (D_\alpha \times \{\eta\})$. Then $\mathscr{C}(\Delta')$ can be identified to the set $\{c(i) \mid i \in \{1, \ldots, r\}\}$. With these notations, the zeta-function formula becomes

$$\zeta_f(t) = \zeta_{f_{|\{l=0\}}}(t) \cdot \prod_{i \in \{1, \ldots, r\}} \zeta_{h^{\mathrm{rel}}_{c(i)}}(t^{n_{i,1}}) \,, \tag{8}$$

where $h^{\mathrm{rel}}_{c(i)}\colon H^\bullet(l_\alpha^{-1}(c(\delta)), l_\alpha^{-1}(a(\delta))) \circlearrowleft$ is the relative monodromy and its zeta-function is $\zeta_{h^{\mathrm{rel}}_{c(i)}}(t) = \zeta_{h'_{c(\delta)}}(t)\zeta^{-1}_{h'_{a(\delta)}}(t)$. One also gets $\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \Sigma_{i\in\{1,\dots,r\},\, n_{i,1}=1}\Lambda(h^{\mathrm{rel}}_{c(i)})$.

By standard arguments, $H^\bullet(l_\alpha^{-1}(c(\delta)), l_\alpha^{-1}(a(\delta)))$ is isomorphic to the direct sum of reduced cohomologies $\oplus_{v\in l_\alpha^{-1}(c(\delta))\cap\Gamma}\tilde{H}^{\bullet-1}(F'_v)$, where $F'_v := B_{v,\varepsilon}\cap l_\alpha^{-1}(a(\delta))$ is the local Milnor fibre and $B_{v,\varepsilon}$ is a Milnor ball of the isolated singularity at v. Let $d_i := \# l_\alpha^{-1}(c(i))\cap\Gamma$.

A point $v\in l_\alpha^{-1}(c(i))\cap\Gamma$ goes, after $n_{i,1}$ complete turns of the carrousel, to $v'\in l_\alpha^{-1}(c(i))\cap\Gamma$ and $v'\neq v$ if $n_{i,1}\geqslant 2$. After exactly $n_{i,1}d_i$ turns, the point v is fixed.

It becomes clear how the relative monodromy acts on the above direct sum; by similar arguments as those in [Si, p. 192], one shows that the matrix of $h^{\mathrm{rel}}_{c(i)}$ may be assumed to have the following block decomposition

$$\begin{bmatrix} 0 & 0 & . & . & . & 0 & \mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i} \\ \mathbf{I} & 0 & . & . & . & 0 & 0 \\ 0 & \mathbf{I} & . & & & . & 0 \\ . & & . & & & . & . \\ . & & & . & & . & . \\ . & & & & . & 0 & . \\ 0 & . & . & . & 0 & \mathbf{I} & 0 \end{bmatrix},$$

where, at some fixed $v(i)\in l_\alpha^{-1}(c(i))$, $\mathbf{I}$ is the identity matrix on $\tilde{H}^\bullet(F'_{v(i)})$, $\mathbf{T}_i$ is the *horizontal monodromy* of the transversal singularity and $\mathbf{V}_i$ is the *vertical monodromy* of the local system on $\Gamma_i\setminus\{0\}$, with fibre $\tilde{H}^\bullet(F'_{v(i)})$. Then $\zeta_{h^{\mathrm{rel}}_{c(i)}}(t) = \Pi_{j\geqslant 0}\det[\mathbf{I} - t^{d_i}\mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i};\ \tilde{H}^j(F'_{v(i)})]^{(-1)^j}$. Finally, our formula looks as follows:

$$\zeta_f(t) = \zeta_{f|\{l=0\}}(t)\cdot\prod_{i\in\{1,\dots,r\}}\prod_{j\geqslant 0}\det[\mathbf{I} - t^{n_{i,1}d_i}\mathbf{V}_i\mathbf{T}_i^{n_{i,1}d_i};\ \tilde{H}^j(F'_{v(i)})]^{(-1)^j}. \tag{9}$$

3.8. This latter one may be easily specialized to the Siersma's formula [*loc. cit.*]. Let A_m be the most exterior annulus and assume that the components of Δ which cut A_m are $\Delta_1,\dots,\Delta_s$ and they have just one Puiseux pair. Denote $D_{m-1} := D_\alpha\times\{\eta\}\setminus A_m$. By our approach we get $\zeta_f(t) = \zeta_{h_{D_{m-1}}}(t)\cdot\Pi_{i\in\{1,\dots,s\}}\zeta_{h^{\mathrm{rel}}_{c(i)}}(t^{n_{i,1}})$.

Let then g be a function with 1-dimensional singular locus $\Sigma = \cup_{i\in\{1,\dots,s\}}\Sigma_i$ and let $f := g + l^K$, for some $l\in\Omega_g$, with $K\in\mathbf{N}$ large enough. Then f is an isolated singularity and, as shown in [Si], one may identify the monodromy of the Milnor fibre F_g to $h_{D_{m-1}}$. The degree of the covering $\Sigma_i\setminus\{0\}\to\Delta_i\setminus\{0\}$ is d_i. Then one gets [Si, p. 183]:

$$\zeta_f(t) = \zeta_g(t)\cdot\prod_{i\in\{1,\dots,s\}}\det[\mathbf{I} - t^{Kd_i}V_i\cdot T_i^{Kd_i}]^{(-1)^{\dim(\mathbf{X},0)}}. \tag{10}$$

3.9. *Example.* Let $\mathbf{X} := \{x^3 + y^4 + z^3 = 0\} \subset \mathbf{C}^3$ and let $f \in \mathbf{m}_{\mathbf{X},0}$ be the function induced by $\tilde{f} \in \mathbf{m}_{\mathbf{C}^3,0}$, $\tilde{f} = x$. Consider the linear function l induced by $\tilde{l} = y$. Then $l \in \Omega_f$. We get that $\Delta(l, f)$ is irreducible and has the Puiseux parametrization: $l = \alpha v^3$, $\lambda = v^4$, where α is a nonzero constant, easy to determine.

Let $c \in \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ and let $a \notin \Delta(l, f) \cap (D_\alpha \times \{\eta\})$ be a neighbour point of c.

The monodromy h'_a can be identified to the monodromy of the function $f_a: (\mathbf{X}_a, 0) \to (\mathbf{C}, 0)$ induced by $\tilde{f}_a = v$, where $\mathbf{X}_a := \{x = v^4, y = v^3, z = \sqrt[3]{2}\gamma v^4\}$ and γ is a 3-root of -1. We get $\zeta_{h'_a}(t) = (1-t)^{-3}$, hence $\zeta_{h_c^{\mathrm{rel}}} = (1-t)^2$.

By using (8), the final result is $\zeta_f(t) = (1-t)^{-3}(1-t^4)^2$. We also get $\Lambda(f) = 3$.

Notice that there is another way of computing the zeta function in this example, by using the usual $\mathbf{C}^*$-action on $\mathbf{X}$, which fixes the zero set $\{\tilde{f} = 0\}$. It follows that the monodromy h_f of f is equal to the 3rd power of the monodromy h_g of the function $g: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $g = y^4 + z^3$ and $\zeta_{h_g^3}(t)$ can be computed from the eigenvalues of h_g. If we change the above function $\tilde{f}$ into $\tilde{f}_1 := x + y$, then the set $\{\tilde{f}_1 = 0\}$ is no more invariant under the above-mentioned $\mathbf{C}^*$-action. The computations for the zeta-function of h_{f_1} are slightly more complicated, since we get two Puiseux pairs, with $n_{1,1} = 1$, $n_{1,2} = 3$. This time, the result is $\zeta_{f_1}(t) = (1-t)^{-1}(1-t^3)^{-1}(1-t^9)$.

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