Of course, one needs a continuous transition between two annuli. The *transition zone* will be a sufficiently thin annulus connecting A_i to A_{i+1}, such that the collection of A_i's and transition zones give a partition of the carrousel disc.

2. Lefschetz number via the carrousel

Let $\mathbf{m}_{\mathbf{X},0}$ denote the maximal ideal of the local ring $\mathcal{O}_{\mathbf{X},0}$. A'Campo proves via the resolution of singularities that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, then $\Lambda(f) = 0$ ([A'C-1, Théorème 1 bis]).

Alternatively, the carrousel construction can provide information on the Lefschetz number. This was the idea of Lê, who showed that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, and $(\mathbf{X}, 0)$ is smooth, then the carrousel monodromy has no fixed points outside the slice $\{l = 0\}$, so $\Lambda(f) = 0$ by induction.

We extend this result by studying the set of fixed points in case $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

2.1. Theorem. *Let all the irreducible components of* $(\mathbf{X}, 0)$ *have dimensions greater than* 1. *If* $n_{i,1} > 1$, $\forall i \in \{1, ..., r\}$, *then* $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

Proof. Assume that $\Delta \not\subset \{u = 0\}$. Since $n_{i,1} > 1$, the carrousel construction tells us that the discs $\delta_{s,j}$ (defined in 1.8), with $n_{s,1} = n_{i,1}$, are cyclically permuted (by a cycle of length $n_{i,1}$).

We may conclude that no point in the carrousel disc is fixed, except the centre and, possibly, some subsets in the transition zones. In each transition zone the subset of fixed points is a finite union of circles, all centred at $(0, \eta)$.

One can decompose the Milnor fibre F_f into suitable pieees on which the geometric monodromy acts and such that the Mayer-Vietoris exact sequence can be applied. Actually, we first cover the carrousel disc by some annuli like those defined in 1.8, then lift this patching to the Milnor fibre. If A_0 is small enough, then $l_\alpha^{-1}(0, \eta)$ is a deformation retract of $l_\alpha^{-1}(A_0)$.

We may conclude: $\Lambda(f) = \Lambda(f_{|\{l=0\}})$, provided that the Lefschetz number of the restriction of the monodromy on any piece of F_f which is the lift by l_α of some pointwise fixed circle is zero. This fact is emphasized in the next lemma, whose proof is left to the reader. The case $\Delta \subset \{u = 0\}$ leads to the same conclusion. □

Lemma. *If the carrousel disc* $D_\alpha \times \{\eta\}$ *contains a circle* S *of fixed points, all of them regular values for the map* l_α, *then* $\Lambda(h_f; H^\bullet(l_\alpha^{-1}(S))) = 0$. □

2.2. *Example.* Let $(\mathbf{X}, 0)$ be a 2-dimensional isolated cyclic quotient singularity, where $\mathbf{X}$ is the algebraic quotient of $\mathbf{C}^2$ by a cyclic group of order 5, usually denoted by $\mathbf{X}_{5,2}$: if ξ is a primitive 5-root of 1, then a generator of our group acts on $\mathbf{C}^2$ by $(x, y) \mapsto (\xi x, \xi^2 y)$.

Let $\tilde{f}: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $\tilde{f} = x^5 + y^5$ and let $f: (\mathbf{X}, 0) \to (\mathbf{C}, 0)$ be the induced function on the quotient. Take a function $\tilde{l}: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $\tilde{l} = xy^2$ and let l be the induced linear function on $(\mathbf{X}, 0)$. Then $l \notin \hat{\Omega}_f$, but $l \in \Omega_f$. Notice that $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

We get that $\Delta'(l, f)$ is irreducible and has a 1-term Puiseux parametrization with Puiseux pair $(3, 5)$. There follows $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

The Milnor fibre of $f_{|\{l=0\}}$ has two components: each of them is the Milnor fibre of a linear function on $(\mathbf{C}, 0)$. This implies that $\Lambda(f_{|\{l=0\}}) = 2$, hence $\Lambda(f) = 2$.

2.3. Corollary [A'C-1, Théorème 1bis]. *Let* $(\mathbf{X}, 0)$ *be an analytic germ of dimension* $\geqslant 1$. *If* $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then* $\Lambda(f) = 0$.

Proof. Let $(\mathbf{X}, 0) = (\mathbf{X}_1, 0) \cup (\mathbf{X}_2, 0)$, where $(\mathbf{X}_2, 0)$ is the union of the irreducible components of $(\mathbf{X}, 0)$ which are of dimension $\geqslant 2$ and $(\mathbf{X}_1, 0)$ is the union of the 1-dimensional branches of $(\mathbf{X}, 0)$.

We slice $(\mathbf{X}_2, 0)$ by a general hyperplane defined by some $l \in \Omega_f$ and treat separately the 1-dimensional components of the slice. If $f \in \mathbf{m}^2_{\mathbf{X}_2,0}$ then each component of the Cerf diagram $\Delta(l, f)$ is tangent to the axis $\{\lambda = 0\}$, provided that l is *general enough*. The proof of this fact is similar to the proof of [Lê-4, Proposition 1.2], but this time the underlying space may be not smooth (see [Ti] for details).

Tangency to $\{\lambda = 0\}$ means exactly that $m_{i,1}/n_{i,1} < 1$, in particular $n_{i,1} > 1$, $\forall i \in \{1, \ldots, r\}$. Thus, our proof relays on a decreasing induction: at each step, we may apply Theorem 2.1. The assertion for 1-dimensional branches is proved by the next easy lemma. $\square$

Lemma. *If* $(\mathbf{X}, 0)$ *is 1-dimensional, irreducible and if* $f \in \mathbf{m}^2_{\mathbf{X},0}$ *then there is a geometric monodromy of* f *without fixed points.* $\square$

As a complement to Theorem 2.1, we have the following precise determination of the Lefschetz number in case $\dim(\mathbf{X}, 0) = 1$:

2.4. Proposition. *If* $(\mathbf{X}, 0) = \bigcup_{i \in R} (C_i, 0)$ *is a curve and its decomposition into irreducible components, then, for any* $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$, *we have:*

$$\Lambda(f) = \# \{ i \in R \mid (C_i, 0) \text{ is smooth and } f \in \mathbf{m}_{C_i,0} \setminus \mathbf{m}^2_{C_i,0} \}.$$

Proof. Let $f_i := f_{|(C_i,0)}$. Then the Milnor fibre of f is a finite set, the disjoint union of the Milnor fibres of f_i, $i \in R$. Hence, $\Lambda(f) = \Sigma_{i\in R}\Lambda(f_i)$.

If $(C_i, 0)$ is smooth, then one has: $\Lambda(f_i) = 1$ if and only if $f_i \in \mathbf{m}_{C_i,0} \setminus \mathbf{m}^2_{C_i,0}$.

If $(C_i, 0)$ is not smooth, let $n_i: (\tilde{C}_i, a_i) \to (C_i, 0)$ be its normalization. It follows $f_i \circ n_i \in \mathbf{m}^2_{\tilde{C}_i,a_i}$, hence the geometric monodromy of f_i is fixed-point-free and $\Lambda(f_i) = 0$. $\square$

2.5. Define $P^{(1)} := \{i \in \{1, \ldots, r\} \mid n_{i,1} = 1\}$.

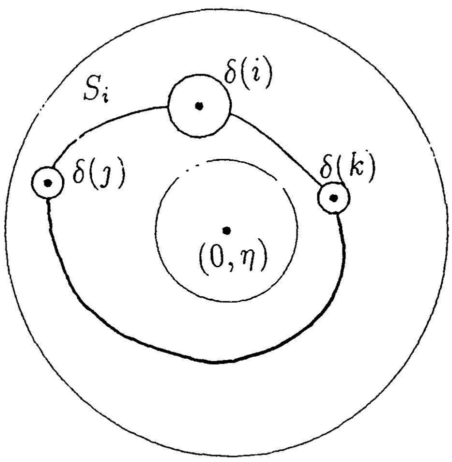
For $i \in P^{(1)}$, let B_i be the union of all carrousel discs of order 1 included in A_i. Then the carrousel construction tells us that the set $A_i \setminus B_i$ is pointwise fixed.

Further, let $\delta(j) \subset A_i$ be a carrousel disc of order 1 defined in the next 2.6. If there are no carrousel discs of order 1 included in $\delta(i)$, then the only fixed point of $\delta(i)$ is its centre. If $\delta(i)$ contains some carrousel disc of order 1 (see Remark 1.6), then we decompose $\delta(i)$ into annuli, since $\delta(i)$ is itself a carrousel. For those annuli that contain some carrousel disc of order 1, we may adapt the present argument, from the beginning of 2.5.

It is easily seen that the set $A_i \setminus B_i$, for $i \in P^{(1)}$, retracts to the subset:

$$(S_i \setminus \bigcup_{\delta \in \mathscr{K}_i^{(1)}} \delta) \cup \bigcup_{\delta \in \mathscr{K}_i^{(1)}} \partial\bar{\delta}\,, \tag{4}$$

where $\mathscr{K}_i^{(1)}$ is the set of carrousel discs of order 1 in A_i which are not included in other carrousel discs of the same order and S_i is a closed curve homotopic to a circle which intersects all the discs $\delta \in \mathscr{K}_i^{(1)}$.



The picture shows a possible shape of the retract of the set of fixed points inside $A_i \setminus B_i$: the "thick" curves are fixed. (The situation in the picture corresponds to $n_{i,1}/m_{i,1} = n_{j,1}/m_{j,1} = n_{k,1}/m_{k,1}$).

Then some neighbourhood of the set of fixed points after one turn of the big carrousel retracts to a set with a finite number of connected components, each of which being either:

(a) a circle centred at $(0, \eta)$ or at a centre of some carrousel disc of order 1, or

(b) a set defined as in (4) or — if case — a similar one in a carrousel disc of order 1, or

(c) a centre of a carrousel disc of order 1 inside A_i, for some $i \in P^{(1)}$, or

(d) the centre $(0, \eta)$ of the big carrousel disc.

2.6. *Definition.* Let $\mathscr{I}^{(1)}$ be a maximal set of indices $i \in P^{(1)}$ such that, if $i_1, i_2 \in \mathscr{I}^{(1)}$, then $\hat{C}^{(1)}_{i_1} \neq \hat{C}^{(1)}_{i_2}$.

For any $i \in \mathscr{I}^{(1)}$, denote by $\delta(i)$ the carrousel disc of order 1 centred at the point $c(i) := \hat{C}^{(1)}_i \cap (D_\alpha \times \{\eta\})$. Let $a(i)$ be an arbitrarily chosen point on the boundary $\partial\bar{\delta}(i)$; it is, by definition, a regular value for l_α.

Definition. Let $a \in (D_\alpha \backslash 0) \times \{\eta\}$ and let F'_a be the fibre of l_α over a. If a is fixed by the carrousel, then the monodromy h_f restricts to an action on $H^\bullet(F'_a)$, denoted by h'_a.

With these notations, we may formulate the following:

2.7. THEOREM. *If* $f \in \mathbf{m}_{X,0}$ *and* $l \in \Omega_f$, *then:*

$$\Lambda(f) = \Lambda(f_{|\{l=0\}}) + \sum_{i \in \mathscr{I}^{(1)}} [\Lambda(h'_{c(i)}) - \Lambda(h'_{a(i)})] .$$

Proof. The Lefschetz number $\Lambda(f)$ splits into a sum, following the decomposition of the set of fixed points into connected components, see 2.5. We use a suitable open covering of a set defined as in (4) and then apply the Mayer-Vietoris exact sequence. The reason of considering the set $\mathscr{I}^{(1)}$ relies on the above discution. By a straightforward computation, using also Lemma 2.1, we get our formula. $\square$

Notice that carrousel discs of order $\geqslant 2$ do not enter in the above formula. For the computation of $\Lambda(h'_{c(i)})$, $\Lambda(h'_{a(i)})$, we refer to Remarks 3.6. There will be an example at the end.

3. ZETA-FUNCTION AND CARROUSEL MONODROMIES

3.1. Loosely speaking, each "important point" of the carrousel disc is fixed after a finite number of turns of the carrousel. We have seen that the set of fixed points after one turn determines the Lefschetz number $\Lambda(h_f)$. So the set of fixed points after k turns is the one responsible for the number $\Lambda(h_f^k)$. It may contain a finite number of circles consisting of regular values for the map l_α. Actually, these circles do not count, as shown by Lemma 2.1 (where