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Artikel: CARROUSEL MONODROMY AND LEFSCHETZ NUMBER OF SINGULARITIES

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The formula for $\zeta_f(t)$ will be not the same, but quite similar to the ones before. The ingredients are zeta-functions of fibres over certain periodic points in the carrousel disc. We show in Sections 2 and 3 how to define these points from the Puiseux expansion of $\Delta(l, f)$. We end by some applications.

Aknowledgement. This work is based on a piece of the author's disertation [Ti]. He much benefited from talks with Dirk Siersma, whose paper [Si] incited him to do this research (see 3.8).

1. THE CARROUSEL REVISITED

1.1. We first briefly recall the carrousel construction, following closely [Lê-1] and [Lê-3], then give the necessary definitions for our study. One regards $(\mathbf{X}, x)$ as being embedded in $(\mathbf{C}^N, 0)$, for some sufficiently large $N \in \mathbf{N}$. We assume that, unless otherwise stated, all the irreducible components of $(\mathbf{X}, 0)$ have dimensions greater than 1.

Let $\mathscr{X}$ be a small enough representative of $(\mathbf{X}, 0)$. Let $\Gamma(l, f)$ be the *polar curve* of f with respect to a linear function $l: (\mathbf{X}, 0) \to (\mathbf{C}, 0)$, relatively to a fixed *Whitney stratification* $\mathscr{S}$ on $\mathscr{X}$ which satisfies *Thom condition* (a_f).

The polar curve $\Gamma(l, f)$ exists for a Zariski open subset $\hat{\Omega}_f$ in the space of linear germs $l: (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$. If one does not impose $\Gamma(l, f)$ to be reduced then one gets a larger set $\Omega_f \supset \hat{\Omega}_f$ which is sometimes useful to deal with (see e.g. Example 2.2). (We only mention that one can enlarge even Ω_f: modify its definition by allowing also nonlinear functions.)

1.2. Let $l \in \Omega_f$ and let $\Phi := (l, f): (\mathbf{X}, 0) \to (\mathbf{C}^2, 0)$. We denote by (u, λ) the pair of coordinates on $\mathbf{C}^2$.

The curve germ (with reduced structure) $\Delta(l, f) := \Phi(\Gamma(l, f))$ is called the *Cerf diagram* (of f with respect to l, relative to $\mathscr{S}$). We shall use the same notation $\Gamma(l, f)$, respectively $\Delta(l, f)$ for suitable representatives of these germs.

There is a fundamental system of "privileged" open polydiscs in $\mathbf{C}^N$, centred at 0, of the form $(D_\alpha \times P_\alpha)_{\alpha \in A}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha \in A}$ of 2-discs at 0 in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in A$, a topological fibration

$$\Phi_\alpha: \mathscr{X} \cap (D_\alpha \times P_\alpha) \cap \Phi^{-1}(D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\}))$$
$$\to D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\}) .$$

Moreover, f induces a topological fibration

$$f_\alpha\colon \mathscr{X} \cap (D_\alpha \times P_\alpha) \cap f^{-1}(D'_\alpha \backslash \{0\}) \to D'_\alpha \backslash \{0\}\,,$$

respectively

$$f'_\alpha\colon \mathscr{X} \cap (\{0\} \times P_\alpha) \cap f^{-1}(D'_\alpha \backslash \{0\}) \to D'_\alpha \backslash \{0\}\,,$$

which is fibre homeomorphic to the Milnor fibration of f, respectively to the Milnor fibration of $f_{|\{l=0\}}$. The disc D'_α has been chosen small enough such that $\Delta(l, f) \cap \partial \overline{D_\alpha} \times D'_\alpha = \varnothing$.

1.3. One can build an integrable smooth vector field on $D_\alpha \times S'_\alpha$ — where S'_α is some circle in D'_α of radius sufficiently close to the radius of $\partial \overline{D'_\alpha}$ — such that, mainly, it is tangent to $\Delta(l, f) \cap (D_\alpha \times S'_\alpha)$ and it lifts the unit vector field of S'_α by the projection $D_\alpha \times S'_\alpha \to S'_\alpha$. Lifting the former vector field by Φ_α and integrating it, one gets a characteristic homeomorphism of the fibration induced by f_α over S'_α, hence a geometric monodromy of the Milnor fibre F_f of f. We call it the (geometric) *carrousel monodromy.*

For some fixed $\eta \in S'_\alpha$, let

$$l_\alpha\colon \mathscr{X} \cap \Phi_\alpha^{-1}(D_\alpha \times \{\eta\}) \to D_\alpha \times \{\eta\} \tag{1}$$

be the restriction of Φ_α and notice that F_f is homeomorphic to $l_\alpha^{-1}(D_\alpha \times \{\eta\})$.

The integration of the vector field on $D_\alpha \times S'_\alpha$ produces a "carrousel" of the disc $D_\alpha \times \{\eta\}$: the trajectory inside $D_\alpha \times S'_\alpha$ of some point $a \in D_\alpha \times \{\eta\}$ projects onto S'_α; one turn around the circle S'_α moves the point a to some other point $a' \in D_\alpha \times \{\eta\}$. By construction, the vector field restricted to $\{0\} \times S'_\alpha$ is the unit vector field of S'_α, hence the centre $(0, \eta)$ of the carroussel disc is indeed fixed; the circle $\partial \overline{D_\alpha} \times \{\eta\}$ is also pointwise fixed.

The distinguished points $\Delta(l, f) \cap D_\alpha \times \{\eta\}$ of the disc have a complex motion around $(0, \eta)$, depending on the Puiseux parametrizations of the branches of Δ which are not included in $\{u = 0\}$. Moreover, these Puiseux expansions determine the motion of any "important" point in the carrousel, as briefly described in the next.

1.4. Our notation is close to the one in [BK].

Let $\Delta := \Delta(l, f)$ and let $\Delta' = \cup_{i \in \{1, \dots, r\}} \Delta_i$ be the union of those irreducible components of Δ which are not included in $\{u = 0\}$.

For $i \in \{1, \dots, r\}$, we consider a Puiseux parametrization of Δ_i with reduced structure:

$$(2)\qquad \begin{cases} \lambda = t^n \\ u = \sum\limits_{j \geqslant m} c_j t^j , & \text{for some } m, n \in \mathbf{Z}_+ , c_j \in \mathbf{C}, c_m \neq 0 . \end{cases}$$

Notice that $m \leqslant n$. The Puiseux parametrization enables one to formally write u as a function of λ:

$$(3)\qquad \begin{aligned} u = {} & a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + a_{k_2}\lambda^{m_2/n_1 n_2} \\ & + \sum_{l=1}^{l_2} b_{2,l}\lambda^{(m_2+l)/n_1 n_2} + \cdots + a_{k_g}\lambda^{m_g/n_1\cdots n_g} + \sum_{l>0} b_{g,l}\lambda^{(m_g+l)/n_1\cdots n_g} , \end{aligned}$$

where g is a positive integer, $\gcd(m_j, n_j) = 1, \forall j \in \{1, \dots,, g\}$ and $n_j \neq 1$, $\forall j \in \{2, \dots, g\}$. Notice that $m_1/n_1 = m/n$ and $a_{k_1} = c_m$.

1.5. We now define two sequences $\{C_i^{(j)}\}_{j \in \{1, \dots g\}}$, $\{\hat{C}_i^{(j)}\}_{j \in \{1, \dots g\}}$ of successive approximation of Δ_i, $i \in \{1, \dots, r\}$:

$$C_i^{(j)}\colon\ u = a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + \cdots + a_{k_j}\lambda^{m_j/n_1\cdots n_j} ,$$

$$\begin{aligned} \hat{C}_i^{(j)}\colon\ u = {} & a_{k_1}\lambda^{m_1/n_1} + \sum_{l=1}^{l_1} b_{1,l}\lambda^{(m_1+l)/n_1} + \cdots + a_{k_j}\lambda^{m_j/n_1\cdots n_j} \\ & + \sum_{l=1}^{l_j} b_{j,l}\lambda^{(m_j+l)/n_1\cdots n_j} \end{aligned}$$

and $\hat{C}_i^{(g)} = \Delta_i$.

The curve $C_i^{(1)}$ intersects the carrousel disc $D_\alpha \times \{\eta\}$ in n_1 points situated on a circle and their carrousel motion is a rotation of angle $2\pi m_1/n_1$. If we take $\hat{C}_i^{(1)}$ instead, we get also n_1 intersection points but their position is a slight perturbation of the previous one.

Each of the points $C_i^{(1)} \cap (D_\alpha \times \{\eta\})$ is the centre of a small disc which contains just one point from the set $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. This latter one, called a *distinguished point,* becomes the centre of a *new (smaller) carrousel.*

Our next definition will play a central role.

1.6. *Definition.* We call *carrousel disc of order k* a sufficiently small open disc centred at some point $c \in \hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, $i \in \{1, \dots, r\}$, which contains all the points $\hat{C}_j^{(k+l)} \cap (D_\alpha \times \{\eta\}), \forall l > 0, \forall j \in \{1, \dots, r\}$ such that $\hat{C}_j^{(k)} = \hat{C}_i^{(k)}$, which are close enough ("satellites") to c. If δ_1, δ_2 are two smaller carrousel discs (not necessarily of the same order), then they are either disjoint or included one in the other.

We may and do assume that the carrousel discs of order k centred at the points $\hat{C}_i^{(k)} \cap (D_\alpha \times \{\eta\})$, $i \in \{1, \ldots, r\}$, are of equal radii.

Remark. A small carrousel disc of order k may contain other carrousel discs of the same order. In the next example:

$$\Delta_1: \quad u_1 = \lambda^{3/2} + \lambda^{17/2}, \qquad C_1^{(1)} \neq \hat{C}_1^{(1)},\ \Delta_1 = \hat{C}_1^{(1)},$$
$$\Delta_2: \quad u_2 = \lambda^{3/2} + \lambda^{7/4}, \qquad C_2^{(1)} = \hat{C}_2^{(1)} = C_1^{(1)},\ \Delta_2 = C_2^{(2)},$$

a carrousel disc of order 1 corresponding to Δ_2 contains a carrousel disc of order 1 corresponding to Δ_1.

1.7. Finally, a simultaneous parametrization of all analytic branches of $\Delta' : \lambda = t^n$, $u_k = \Sigma_{j \geqslant m_k} a_{k,j} t^j$, for $k \in \{1, \ldots, r\}$, leads to the construction of the full carrousel.

If we define the "essential" curve associated to Δ_i by:

$$\Delta_i^{\mathrm{es}}: \ u = a_{k_1} \lambda^{m_1/n_1} + a_{k_2} \lambda^{m_2/n_1 n_2} + \cdots + a_{k_g} \lambda^{m_g/n_1 \cdots n_g},$$

then the carrousel associated to $\Delta^{\mathrm{es}} = \bigcup_{i \in \{1, \ldots, r\}} \Delta_i^{\mathrm{es}}$ might be called an "ideal carrousel". However, the topological type of the link Δ' may be *not* the same as the one of Δ^{es}.

1.8. Denote by $(m_{i,j}, n_{i,j})_{j \in \{1, \ldots, g_i\}}$ the Puiseux pairs of Δ_i, $\forall i \in \{1, \ldots, r\}$. Suppose that we have the following ordering among the first Puiseux pairs (eventually after some permutation of indices): $m_{1,1}/n_{1,1} \geqslant m_{2,1}/n_{2,1} \geqslant \cdots \geqslant m_{r,1}/n_{r,1}$.

To each branch Δ_i there corresponds an annulus A_i — with central symmetry at $(0, \eta)$ — inside the carrousel disc, such that A_i contains $\Delta_i \cap (D_\alpha \times \{\eta\})$, see [Lê-1]. We also define A_0 to be an arbitrarily small open disc centred in $(0, \eta)$. By definition, $A_i = A_j$ if and only if $m_{i,1}/n_{i,1} = m_{j,1}/n_{j,1}$.

For any $i \in \{1, \ldots, r\}$, there are $n_{i,1}$ carrousel discs $\delta_{i,j}$, $j \in \{1, \ldots, n_{i,1}\}$, of order 1, centred at the $n_{i,1}$ points $\hat{C}_i^{(1)} \cap (D_\alpha \times \{\eta\})$. In case of the "ideal" carrousel, these points rotate around $(0, \eta)$ by $2\pi m_{i,1}/n_{i,1}$. The annulus A_i contains all the carrousel discs $\delta_{s,j}$ such that $C_s^{(1)} = C_i^{(1)}$. Each point of the annulus A_i, outside any disc $\delta_{s,j}$, is fixed by the $n_{i,1}$th iterate of the carrousel. The disc A_0 is just pointwise fixed by the carrousel.

Of course, one needs a continuous transition between two annuli. The *transition zone* will be a sufficiently thin annulus connecting A_i to A_{i+1}, such that the collection of A_i's and transition zones give a partition of the carrousel disc.

2. Lefschetz number via the carrousel

Let $\mathbf{m}_{\mathbf{X},0}$ denote the maximal ideal of the local ring $\mathcal{O}_{\mathbf{X},0}$. A'Campo proves via the resolution of singularities that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, then $\Lambda(f) = 0$ ([A'C-1, Théorème 1bis]).

Alternatively, the carrousel construction can provide information on the Lefschetz number. This was the idea of Lê, who showed that, if $f \in \mathbf{m}^2_{\mathbf{X},0}$, and $(\mathbf{X}, 0)$ is smooth, then the carrousel monodromy has no fixed points outside the slice $\{l = 0\}$, so $\Lambda(f) = 0$ by induction.

We extend this result by studying the set of fixed points in case $f \in \mathbf{m}_{\mathbf{X},0} \setminus \mathbf{m}^2_{\mathbf{X},0}$.

2.1. Theorem. *Let all the irreducible components of* $(\mathbf{X}, 0)$ *have dimensions greater than* 1. *If* $n_{i,1} > 1$, $\forall i \in \{1, \ldots, r\}$, *then* $\Lambda(f) = \Lambda(f_{|\{l=0\}})$.

Proof. Assume that $\Delta \not\subset \{u = 0\}$. Since $n_{i,1} > 1$, the carrousel construction tells us that the discs $\delta_{s,j}$ (defined in 1.8), with $n_{s,1} = n_{i,1}$, are cyclically permuted (by a cycle of length $n_{i,1}$).

We may conclude that no point in the carrousel disc is fixed, except the centre and, possibly, some subsets in the transition zones. In each transition zone the subset of fixed points is a finite union of circles, all centred at $(0, \eta)$.

One can decompose the Milnor fibre F_f into suitable pieces on which the geometric monodromy acts and such that the Mayer-Vietoris exact sequence can be applied. Actually, we first cover the carrousel disc by some annuli like those defined in 1.8, then lift this patching to the Milnor fibre. If A_0 is small enough, then $l_\alpha^{-1}(0, \eta)$ is a deformation retract of $l_\alpha^{-1}(A_0)$.

We may conclude: $\Lambda(f) = \Lambda(f_{|\{l=0\}})$, provided that the Lefschetz number of the restriction of the monodromy on any piece of F_f which is the lift by l_α of some pointwise fixed circle is zero. This fact is emphasized in the next lemma, whose proof is left to the reader. The case $\Delta \subset \{u = 0\}$ leads to the same conclusion. □

Lemma. *If the carrousel disc* $D_\alpha \times \{\eta\}$ *contains a circle* S *of fixed points, all of them regular values for the map* l_α, *then* $\Lambda(h_f; H^\bullet(l_\alpha^{-1}(S))) = 0$. □