Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 39 (1993)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CARROUSEL MONODROMY AND LEFSCHETZ NUMBER OF SINGULARITIES

Autor: Tibr, Mihai

Kapitel: Introduction

DOI: https://doi.org/10.5169/seals-60425

L'Enseignement Mathématique, t. 39 (1993), p. 233-247

CARROUSEL MONODROMY AND LEFSCHETZ NUMBER OF SINGULARITIES

by Mihai TIBĂR

INTRODUCTION

Let $f: (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be a holomorphic function on an analytic germ $(\mathbf{X}, x)$. Let h_f denote the monodromy of the germ $\Psi_f^{\bullet}(\mathbf{C}_{\mathbf{X}}^{\bullet})_x$ of neighbouring cycles. One defines its *Lefschetz number*

$$\Lambda(h_f) := \sum_{i \geqslant 0} (-1)^i \text{ trace } [h_f; \Psi_f^i(\mathbf{C}_{\mathbf{X}}^{\bullet})_x] ,$$

and its *zeta-function*

$$\zeta_{h_f}(t) := \prod_{i \geqslant 0} \det[I - t \cdot h_f ; \Psi_f^i(\mathbf{C}_{\mathbf{X}}^{\bullet})_x]^{(-1)^{i+1}} .$$

We alternatively denote them by $\Lambda(f)$, respectively $\zeta_f(t)$.

A theorem of Eisenbud and Neumann [EN, Theorem 4.3] asserts that the zeta-function of a *fibred multilink* L is the product of the zeta-functions over all *splice components* of L. If the multilink is defined by some Cerf diagram $\Delta(l, f)$, then $\zeta_f(t)$ becomes the zeta-function of the multilink L, this time with coefficients in a local system. This observation of Némethi [Ne] enables him to prove an inductive formula for $\zeta_f(t)$, in terms of invariants of the so called EN-diagram (splice diagram); compare to the one of Eisenbud and Neumann [EN, p. 96]. Some quite strong results in the 3-dimensional link theory are involved in the proofs.

Our approach is based on Lê's carrousel construction and is therefore more geometric and selfcontained. It yields inductive formulae for $\Lambda(f)$ and $\zeta_f(t)$ directly from the Puiseux parametrization of $\Delta(l, f)$. Moreover, it clarifies the contribution, however essential in general, of the "nonessential" terms in this parametrization — which may be not clear from the definition of the splice diagram of an algebraic link given in [EN, p. 53], simply because such terms are completely omitted. One can therefore compare to our definitions 1.5 ÷ 7.

The formula for $\zeta_f(t)$ will be not the same, but quite similar to the ones before. The ingredients are zeta-functions of fibres over certain periodic points in the carrousel disc. We show in Sections 2 and 3 how to define these points from the Puiseux expansion of $\Delta(l, f)$. We end by some applications.

Aknowledgement. This work is based on a piece of the author's disertation [Ti]. He much benefited from talks with Dirk Siersma, whose paper [Si] incited him to do this research (see 3.8).

1. THE CARROUSEL REVISITED

1.1. We first briefly recall the carrousel construction, following closely [Lê-1] and [Lê-3], then give the necessary definitions for our study. One regards $(\mathbf{X}, x)$ as being embedded in $(\mathbf{C}^N, 0)$, for some sufficiently large $N \in \mathbf{N}$. We assume that, unless otherwise stated, all the irreducible components of $(\mathbf{X}, 0)$ have dimensions greater than 1.

Let $\mathscr{X}$ be a small enough representative of $(\mathbf{X}, 0)$. Let $\Gamma(l, f)$ be the *polar curve* of f with respect to a linear function $l: (\mathbf{X}, 0) \to (\mathbf{C}, 0)$, relatively to a fixed *Whitney stratification* $\mathscr{S}$ on $\mathscr{X}$ which satisfies *Thom condition* (a_f).

The polar curve $\Gamma(l, f)$ exists for a Zariski open subset $\hat{\Omega}_f$ in the space of linear germs $l: (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$. If one does not impose $\Gamma(l, f)$ to be reduced then one gets a larger set $\Omega_f \supset \hat{\Omega}_f$ which is sometimes useful to deal with (see e.g. Example 2.2). (We only mention that one can enlarge even Ω_f: modify its definition by allowing also nonlinear functions.)

1.2. Let $l \in \Omega_f$ and let $\Phi := (l, f): (\mathbf{X}, 0) \to (\mathbf{C}^2, 0)$. We denote by (u, λ) the pair of coordinates on $\mathbf{C}^2$.

The curve germ (with reduced structure) $\Delta(l, f) := \Phi(\Gamma(l, f))$ is called the *Cerf diagram* (of f with respect to l, relative to $\mathscr{S}$). We shall use the same notation $\Gamma(l, f)$, respectively $\Delta(l, f)$ for suitable representatives of these germs.

There is a fundamental system of "privileged" open polydiscs in $\mathbf{C}^N$, centred at 0, of the form $(D_\alpha \times P_\alpha)_{\alpha \in A}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha \in A}$ of 2-discs at 0 in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in A$, a topological fibration

$$\Phi_\alpha: \mathscr{X} \cap (D_\alpha \times P_\alpha) \cap \Phi^{-1}(D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\}))$$
$$\to D_\alpha \times D'_\alpha \setminus (\Delta(l, f) \cup \{\lambda = 0\}) .$$