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2. THE MANIFOLDS $M(k, l, m)$

Gompf studied in [G] smooth simply connected minimal elliptic surfaces without multiple fibers. These are classified up to diffeomorphism by a positive integer. For each $n > 0$ there is such a manifold $V_n \to \mathbf{CP}^1$ with no multiple fibers, $6n$ cusp fibers and a section with self-intersection $-n$ which hits each fiber transversally once. Gompf defines N_n, the "nucleus" of V_n, to be a regular neighborhood of a cusp fiber together with the section. Performing a "differentiable logarithmic transform" of multiplicity $p \geqslant 0$ inside N_n gives an elliptic surface $V_n(p)$ with a multiple fiber, and corresponding nucleus $N_n(p)$. In [G] it is proved that $V_n(0)$ decomposes as a connected sum of $\pm \mathbf{CP}^2$, and how this implies that for any fixed odd $n \geqslant 3$ $N_n(0)$ and N_n are homeomorphic but non-diffeomorphic. The manifolds $N_n(0)$ and N_n have the following handlebody descriptions:

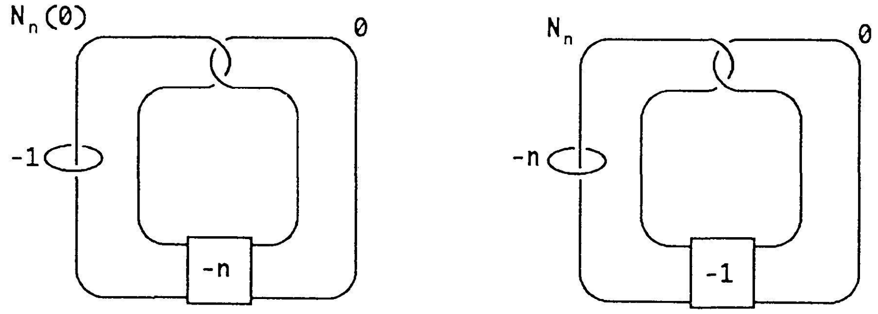

FIGURE 3

As pointed out in [G], these examples can be generalized. In fact, let k, l, m be positive integers, and $M(k, l, m)$ the manifold described in the introduction. The intersection form of $M(k, l, m)$ is $\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$, so $\partial M(k, l, m)$ is a homology sphere. Moreover $\partial M(k, l, m)$ is independent of the order of k, l, m (see below). Hence by [F] and the classification of the symmetric unimodular forms, if either $s > 0$ or $s = 0$ and $k \equiv l \equiv m \bmod 2$, the homeomorphism type of $M(k, l, m) \overset{s}{\#} \overline{\mathbf{CP}}^2$ is independent of the order of k, l, m. Analogously, it is possible to permute any pair of integers with the same parity without altering the homeomorphism type of $M(k, l, m)$. So, for instance, $M(1, n, 1)$ and $M(n, 1, 1)$ are homeomorphic for n odd, and since clearly $N_n(0) = M(1, n, 1)$ and $N_n = M(n, 1, 1)$, they are not diffeomorphic for $n \geqslant 3$.

Let us prove that $\partial M(k, l, m)$ is independent of the order of k, l, m. As observed in the introduction, $M(k, l, m) = M(k, m, l)$ so it is enough to see that $\partial M(k, l, m) = \partial M(m, l, k)$. In fact, $\partial M(k, l, m)$ is given by the first diagram of figure 4 after cancelling the 1-2 pairs, and similarly $\partial M(m, l, k)$ is given by the second one. Moreover, there is an obvious isotopy between the two diagrams.

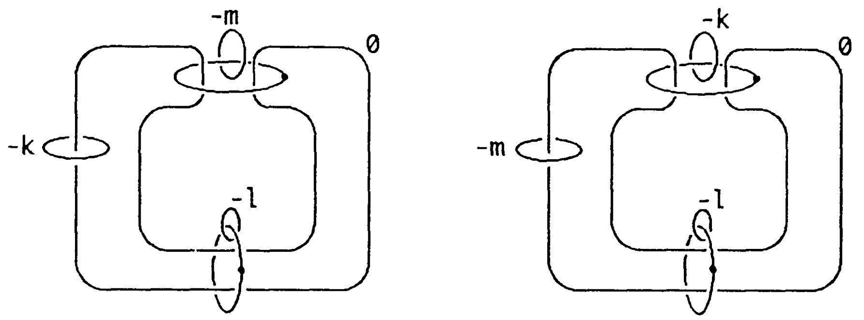

FIGURE 4

We introduce now an auxiliary result, needed for the proof of theorem 1.1. If $q: \Lambda \to \mathbf{Z}$ is a quadratic form on the lattice Λ, and if $\alpha \in \Lambda$, $q(\alpha) = -1$, then

$$R_\alpha(x) = x + 2(x \,.\, \alpha)\alpha$$

is an integral isometry of Λ. Similarly, if $q(\alpha) = -2$, let

$$R_\alpha(x) = x + (x \,.\, \alpha)\alpha \,.$$

Given an oriented 4-manifold M, let $q_M: H^2(M) \to \mathbf{Z}$ its intersection form. Then there is the following elementary fact:

PROPOSITION 2.1 (2.4, chapter III in [FM1]). *Let M be an oriented 4-manifold and $S^2 \subseteq M$ be an embedded sphere with $\alpha \in H^2(M; \mathbf{Z})$ the cohomology class dual to S^2. If $q_M(\alpha) = -1$ or -2, there is an orientation-preserving diffeomorphism φ of M such that $\varphi^* = R_\alpha$.*

We shall use the notations $\hat{N}_k = N_k \overset{l+m-2}{\#} \overline{\mathbf{CP}}^2$, $\bar{N}_k = \hat{N}_k \overset{s}{\#} \overline{\mathbf{CP}}^2$, $\hat{V}_k = V_k \overset{l+m-2}{\#} \overline{\mathbf{CP}}^2$, $\bar{V}_k = \hat{V}_k \overset{s}{\#} \overline{\mathbf{CP}}^2$, and $\bar{M}(k, l, m) = M(k, l, m) \overset{s}{\#} \overline{\mathbf{CP}}^2$.

LEMMA 2.2. *For all $k, l, m > 0$ there is a smooth embedding $i: M(k, l, m) \hookrightarrow \hat{N}_k$, such that $i_* H_2(M(k, l, m)) \subseteq H_2(N_k) \subseteq H_2(\hat{N}_k)$.*

Proof. Blow-up $l + m - 2$ — 1's around the 0-framed knot in a link picture for N_k to get a framed link containing figure 1 plus algebraically unlinked — 1-framed unknots. QED

Remark 2.3. Since $N_k \subset V_k$, by the lemma

$$M(k, l, m) \hookrightarrow \hat{N}_k \subset \hat{V}_k, \quad \text{and} \quad \bar{M}(k, l, m) \hookrightarrow \bar{V}_k .$$

It is easy to see from the proof of the lemma that the image of $H_2(M(k, l, m))$ under the above embeddings is orthogonal to the Poincaré duals of the exceptional classes, and it contains the class $[f]$ of a smooth fiber of the elliptic fibration on $\hat{V}_k$. In fact $[f]$ comes from the class given by the 2-handle attached to the 0-framed component in the link picture (see [G]).

THEOREM 2.4. *Let* l, m *be positive integers with* $l > 2$, *and* s *a non-negative integer. If either* (i) $s \neq 0$ *or* (ii) $s = 0$ *and* $l \equiv 0 \bmod 2$, *the manifolds* $\bar{M}(2, l, m)$ *and* $\bar{M}(l, 2, m)$ *are homeomorphic but their interiors are not diffeomorphic.*

Proof. Suppose we are in case (i). Then, as pointed out above, $\bar{M}(2, l, m)$ and $\bar{M}(l, 2, m)$ are homeomorphic for any l. $\bar{M}(2, l, m)$ contains an embedded sphere S^2 of square -2: take a slicing disk for the -2-framed unknot in the link picture for $M(2, l, m)$ union the core of the corresponding 2-handle. By contradiction, suppose $\psi\colon \operatorname{int}(\bar{M}(2, l, m)) \cong \operatorname{int}(\bar{M}(l, 2, m))$ is a diffeomorphism. By remark 2.3 $\bar{M}(l, 2, m) \subset \bar{V}_l$, so $\psi(S^2) \subset \bar{V}_l$ is an embedded sphere. If ψ were orientation-reversing $\psi(S^2)$ would have positive self-intersection. But for $l \geqslant 2$ $\bar{V}_l$ does not contain such embedded spheres [FM2], hence ψ has to be orientation-preserving. So $\psi(S^2)$ has self-intersection -2. Let $\alpha \in H^2(\bar{V}_l)$ be the Poincaré dual of the homology class carried by $\psi(S^2)$. By proposition 2.1 there is an orientation-preserving self-diffeomorphism φ of $\bar{V}_l$ such that $\varphi^*(\xi) = \xi + (\xi, \alpha)\alpha$ for any $\xi \in H^2(\bar{V}_l)$. This implies that α is orthogonal to the exceptional classes of $\bar{V}_l$, because by [FM2] if e is an exceptional class then $\varphi^*(e) = e$. Therefore $[\psi(S^2)] \in H_2(M(l, 2, m)) \subset H_2(\hat{V}_l)$. Observe that $M(l, 2, m)$ has intersection form $\begin{pmatrix} 0 & 1 \\ 1 & -l \end{pmatrix}$, where the class of square zero may be taken to be the class of a smooth fiber f, hence $[\psi(S^2)] \cdot [f] \neq 0$. Hence $\alpha \cdot k_{\bar{V}_l} \neq 0$, where $k_{\bar{V}_l}$ is the canonical class. Again by [FM2] any orientation-preserving self-diffeomorphism of $\bar{V}_l$ preserves the canonical class k_{V_l} of V_l up to sign and the exceptional classes up to permutation and signs. But since α is not a multiple of k_{V_l}, φ cannot have this property. This gives a contradiction to the existence of ψ and proves the theorem in case (i). To prove the statement in case (ii) it is enough to observe that since, by what we have proved already, the interiors of $M(2, l, m) \# \overline{\mathbf{CP}}^2$ and $M(l, 2, m) \# \overline{\mathbf{CP}}^2$ are not diffeomorphic, the interiors of $M(2, l, m)$ and $M(l, 2, m)$ cannot be diffeomorphic as well. QED

THEOREM 2.5. *Let l, m be positive integers with $l > 1$, and s a non-negative integer. If either* (i) $s \neq 0$ *or* (ii) $s = 0$ *and* $l \equiv 1 \bmod 2$, *then the manifolds $\bar{M}(1, l, m)$ and $\bar{M}(l, 1, m)$ are homeomorphic but their interiors are not diffeomorphic.*

Proof. Use a sphere of square -1 inside $\bar{M}(1, l, m)$ to get a contradiction to the existence of a diffeomorphism exactly as in the proof of theorem 2.4. QED

Proof of theorem 1.1. Put theorems 2.4 and 2.5 together. QED

Now we give our proof of theorem 1.2.

Proof of theorem 1.2. In [A] it is proved that $-W_1$ and $-W_2$ are homeomorphic. To prove that they are not diffeomorphic, notice first that W_1 can be embedded inside $\hat{V}_k = V_k \# 2\overline{\mathbf{CP}}^2$ for any $k \geqslant 1$: blow-up two -1's around the 0-framed knot in a link picture for N_k. This embedding sends a generator of $H_2(W_1)$ to $f + e \in H_2(\hat{V}_k)$, where f is the class of a smooth fiber, and e is the Poincaré dual of an exceptional class. Next observe that a generator of $H_2(W_2)$ is representable by a sphere S^2 smoothly embedded in $\operatorname{int}(W_2)$ with self-intersection -1: the knot K_2 is ribbon, hence slice, so take as S^2 a slicing disk union the core of the corresponding 2-handle. Finally suppose, arguing by contradiction, that $\psi: \operatorname{int}(W_2) \cong \operatorname{int}(W_1)$ is a diffeomorphism. The class $f + e$ is therefore representable by the smoothly embedded sphere $\psi(S^2)$ inside $\hat{V}_k$. Since $\hat{V}_k$ does not contain embedded spheres with positive self-intersection [FM2], we may assume that ψ is orientation preserving and $\psi(S^2)$ has self-intersection -2. Again by [FM2], V_k has big diffeomorphism group with respect to the canonical class, and it has some non-zero generalized Donaldson invariant. Moreover $b_2^+(\hat{V}_k) \geqslant 3$ for $k \geqslant 2$. So when $k \geqslant 2$ we may apply theorem 3.5 from [FM2], which implies that any class of square -1 in $H_2(\hat{V}_k)$ representable by a smoothly embedded 2-sphere is equal, up to sign, to the Poincaré dual of one of the exceptional classes. This is clearly not the case for $f + e$, so we get a contradiction to the existence of ψ. QED