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## SMOOTHLY EMBEDDED 2-SPHERES AND EXOTIC 4-MANIFOLDS

by Paolo LISCA

### 1. INTRODUCTION

We prove two independent results concerning multiple smooth structures on certain topological 4-manifolds with boundary (theorems 1.1 and 1.2). The second of our results has been already proven by Akbulut [A]; we give here a different proof. The common underlying theme consists of obstructions to represent 2-homology classes of 4-manifolds by smoothly embedded spheres. Indeed, we use the fact that certain classes of self-intersection  $-1$  or  $-2$  cannot be represented by smoothly embedded spheres inside certain 4-manifolds, while they can inside others.

With our first result we give a partial answer to the following question, raised in [G]: given the 4-manifold with boundary  $M(k, l, m)$ , obtained by attaching 2-handles to  $B^4$  according to the framed link of figure 1, how many diffeomorphism types are realized by permuting the integers  $k$ ,  $l$ , and  $m$ ?

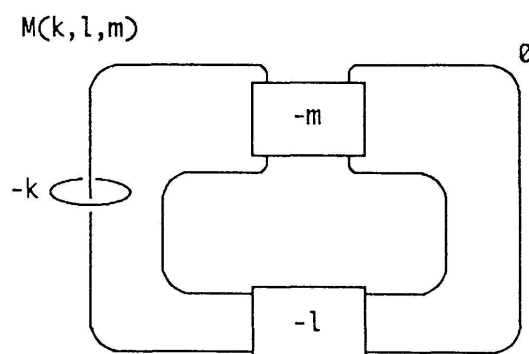


FIGURE 1

(In such pictures, a box with  $-m$  inside means  $-m$  full twists. A  $-k$  next to a loop means that the framing has linking number  $-k$  with the loop.)

Notice the equality  $M(k, l, m) = M(k, m, l)$ , obtained by an obvious isotopy of the link. In section 2 we prove the following:

**THEOREM 1.1.** *Let  $k, l, m, k \neq l$ , be positive integers with either  $k \leq 2$  or  $l \leq 2$ , and  $s$  a non-negative integer. If either (i)  $s \neq 0$  or (ii)  $s = 0$  and  $k \equiv l \pmod{2}$ , then the manifolds  $M(k, l, m) \#^s \overline{\mathbf{CP}}^2$  and  $M(l, k, m) \#^s \overline{\mathbf{CP}}^2$  are homeomorphic but their interiors are not diffeomorphic.*

We have as a corollary that for  $n$  odd  $M(1, n, 1)$  and  $M(n, 1, 1)$  are homeomorphic but not diffeomorphic, a result already obtained in [G] by a different method.

As for the second result, let  $W_i, i = 1, 2$  be the 4-manifolds with boundary obtained via the framed knots  $K_i, i = 1, 2$  of figure 2.

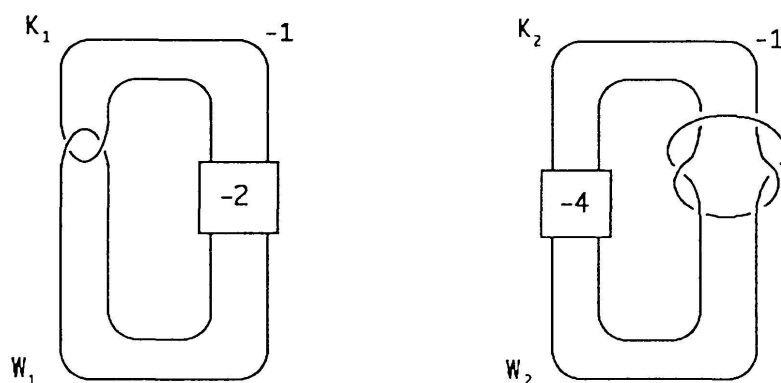


FIGURE 2

We give a new proof of the following theorem, due to Akbulut [A]:

**THEOREM 1.2.**  *$W_1$  and  $W_2$  are homeomorphic but their interiors are not diffeomorphic.*

*Remark.* In Akbulut's notation, the manifolds  $W_1$  and  $W_2$  are respectively  $-Q_1$  and  $-Q_2$ .

To prove theorems 1.1 and 1.2 we use gauge-theoretical results about the smooth structure of blown-up elliptic surfaces [FM2]. We remark that also the analogous results from [G] and [A] depend on gauge theory.

*Acknowledgments.* We thank John Morgan for numerous discussions about embedded spheres in 4-manifolds, and for making available an early copy of his book with Bob Friedman [FM2]. We are also grateful to Riccardo Benedetti for helpful conversations.

2. THE MANIFOLDS  $M(k, l, m)$ 

Gompf studied in [G] smooth simply connected minimal elliptic surfaces without multiple fibers. These are classified up to diffeomorphism by a positive integer. For each  $n > 0$  there is such a manifold  $V_n \rightarrow \mathbf{CP}^1$  with no multiple fibers,  $6n$  cusp fibers and a section with self-intersection  $-n$  which hits each fiber transversally once. Gompf defines  $N_n$ , the “nucleus” of  $V_n$ , to be a regular neighborhood of a cusp fiber together with the section. Performing a “differentiable logarithmic transform” of multiplicity  $p \geq 0$  inside  $N_n$  gives an elliptic surface  $V_n(p)$  with a multiple fiber, and corresponding nucleus  $N_n(p)$ . In [G] it is proved that  $V_n(0)$  decomposes as a connected sum of  $\pm \mathbf{CP}^2$ , and how this implies that for any fixed odd  $n \geq 3$   $N_n(0)$  and  $N_n$  are homeomorphic but non-diffeomorphic. The manifolds  $N_n(0)$  and  $N_n$  have the following handlebody descriptions:

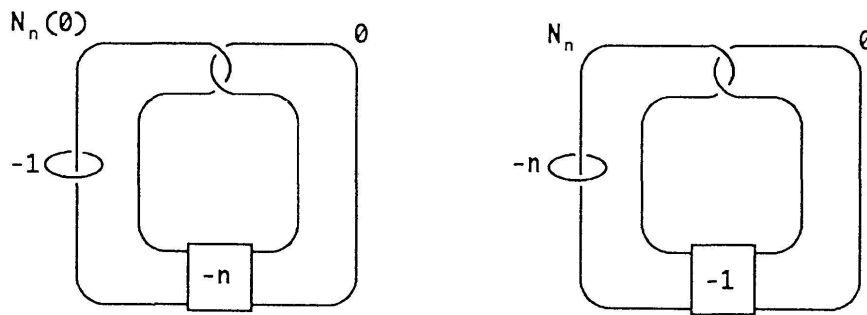


FIGURE 3

As pointed out in [G], these examples can be generalized. In fact, let  $k, l, m$  be positive integers, and  $M(k, l, m)$  the manifold described in the introduction. The intersection form of  $M(k, l, m)$  is  $\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$ , so  $\partial M(k, l, m)$  is a homology sphere. Moreover  $\partial M(k, l, m)$  is independent of the order of  $k, l, m$  (see below). Hence by [F] and the classification of the symmetric unimodular forms, if either  $s > 0$  or  $s = 0$  and  $k \equiv l \equiv m \pmod{2}$ , the homeomorphism type of  $M(k, l, m) \#^s \overline{\mathbf{CP}}^2$  is independent of the order of  $k, l, m$ . Analogously, it is possible to permute any pair of integers with the same parity without altering the homeomorphism type of  $M(k, l, m)$ . So, for instance,  $M(1, n, 1)$  and  $M(n, 1, 1)$  are homeomorphic for  $n$  odd, and since clearly  $N_n(0) = M(1, n, 1)$  and  $N_n = M(n, 1, 1)$ , they are not diffeomorphic for  $n \geq 3$ .

Let us prove that  $\partial M(k, l, m)$  is independent of the order of  $k, l, m$ . As observed in the introduction,  $M(k, l, m) = M(k, m, l)$  so it is enough to see that  $\partial M(k, l, m) = \partial M(m, l, k)$ . In fact,  $\partial M(k, l, m)$  is given by the first diagram of figure 4 after cancelling the 1-2 pairs, and similarly  $\partial M(m, l, k)$  is given by the second one. Moreover, there is an obvious isotopy between the two diagrams.

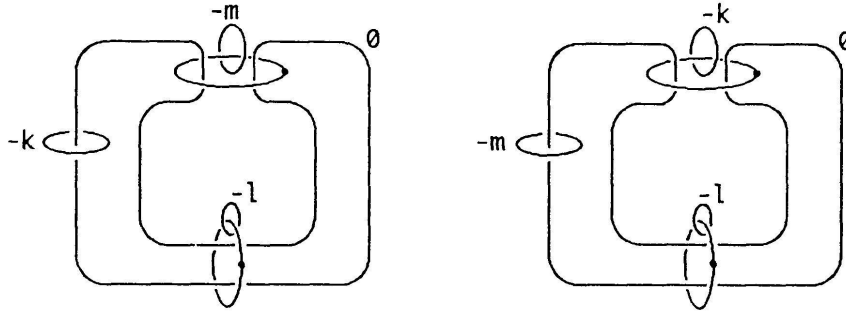


FIGURE 4

We introduce now an auxiliary result, needed for the proof of theorem 1.1. If  $q: \Lambda \rightarrow \mathbf{Z}$  is a quadratic form on the lattice  $\Lambda$ , and if  $\alpha \in \Lambda$ ,  $q(\alpha) = -1$ , then

$$R_\alpha(x) = x + 2(x \cdot \alpha)\alpha$$

is an integral isometry of  $\Lambda$ . Similarly, if  $q(\alpha) = -2$ , let

$$R_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

Given an oriented 4-manifold  $M$ , let  $q_M: H^2(M) \rightarrow \mathbf{Z}$  its intersection form. Then there is the following elementary fact:

**PROPOSITION 2.1** (2.4, chapter III in [FM1]). *Let  $M$  be an oriented 4-manifold and  $S^2 \subseteq M$  be an embedded sphere with  $\alpha \in H^2(M; \mathbf{Z})$  the cohomology class dual to  $S^2$ . If  $q_M(\alpha) = -1$  or  $-2$ , there is an orientation-preserving diffeomorphism  $\phi$  of  $M$  such that  $\phi^* = R_\alpha$ .*

We shall use the notations  $\hat{N}_k = N_k \#^{l+m-2} \overline{\mathbf{CP}}^2$ ,  $\bar{N}_k = \hat{N}_k \#^s \overline{\mathbf{CP}}^2$ ,  $\hat{V}_k = V_k \#^{l+m-2} \overline{\mathbf{CP}}^2$ ,  $\bar{V}_k = \hat{V}_k \#^s \overline{\mathbf{CP}}^2$ , and  $\bar{M}(k, l, m) = M(k, l, m) \#^s \overline{\mathbf{CP}}^2$ .

**LEMMA 2.2.** *For all  $k, l, m > 0$  there is a smooth embedding  $i: M(k, l, m) \hookrightarrow \hat{N}_k$ , such that  $i_* H_2(M(k, l, m)) \subseteq H_2(N_k) \subseteq H_2(\hat{N}_k)$ .*

*Proof.* Blow-up  $l + m - 2 - 1$ 's around the 0-framed knot in a link picture for  $N_k$  to get a framed link containing figure 1 plus algebraically unlinked  $-1$ -framed unknots. QED

*Remark 2.3.* Since  $N_k \subset V_k$ , by the lemma

$$M(k, l, m) \hookrightarrow \hat{N}_k \subset \hat{V}_k, \quad \text{and} \quad \bar{M}(k, l, m) \hookrightarrow \bar{V}_k.$$

It is easy to see from the proof of the lemma that the image of  $H_2(M(k, l, m))$  under the above embeddings is orthogonal to the Poincaré duals of the exceptional classes, and it contains the class  $[f]$  of a smooth fiber of the elliptic fibration on  $\hat{V}_k$ . In fact  $[f]$  comes from the class given by the 2-handle attached to the 0-framed component in the link picture (see [G]).

**THEOREM 2.4.** *Let  $l, m$  be positive integers with  $l > 2$ , and  $s$  a non-negative integer. If either (i)  $s \neq 0$  or (ii)  $s = 0$  and  $l \equiv 0 \pmod{2}$ , the manifolds  $\bar{M}(2, l, m)$  and  $\bar{M}(l, 2, m)$  are homeomorphic but their interiors are not diffeomorphic.*

*Proof.* Suppose we are in case (i). Then, as pointed out above,  $\bar{M}(2, l, m)$  and  $\bar{M}(l, 2, m)$  are homeomorphic for any  $l$ .  $\bar{M}(2, l, m)$  contains an embedded sphere  $S^2$  of square  $-2$ : take a slicing disk for the  $-2$ -framed unknot in the link picture for  $M(2, l, m)$  union the core of the corresponding 2-handle. By contradiction, suppose  $\psi: \text{int}(\bar{M}(2, l, m)) \cong \text{int}(\bar{M}(l, 2, m))$  is a diffeomorphism. By remark 2.3  $\bar{M}(l, 2, m) \subset \bar{V}_l$ , so  $\psi(S^2) \subset \bar{V}_l$  is an embedded sphere. If  $\psi$  were orientation-reversing  $\psi(S^2)$  would have positive self-intersection. But for  $l \geq 2$   $\bar{V}_l$  does not contain such embedded spheres [FM2], hence  $\psi$  has to be orientation-preserving. So  $\psi(S^2)$  has self-intersection  $-2$ . Let  $\alpha \in H^2(\bar{V}_l)$  be the Poincaré dual of the homology class carried by  $\psi(S^2)$ . By proposition 2.1 there is an orientation-preserving self-diffeomorphism  $\phi$  of  $\bar{V}_l$  such that  $\phi^*(\xi) = \xi + (\xi, \alpha)\alpha$  for any  $\xi \in H^2(\bar{V}_l)$ . This implies that  $\alpha$  is orthogonal to the exceptional classes of  $\bar{V}_l$ , because by [FM2] if  $e$  is an exceptional class then  $\phi^*(e) = e$ . Therefore  $[\psi(S^2)] \in H_2(M(l, 2, m)) \subset H_2(\hat{V}_l)$ . Observe that  $M(l, 2, m)$  has intersection form  $\begin{pmatrix} 0 & 1 \\ 1 & -l \end{pmatrix}$ , where the class of square zero may be taken to be the class of a smooth fiber  $f$ , hence  $[\psi(S^2)] \cdot [f] \neq 0$ . Hence  $\alpha \cdot k_{\bar{V}_l} \neq 0$ , where  $k_{\bar{V}_l}$  is the canonical class. Again by [FM2] any orientation-preserving self-diffeomorphism of  $\bar{V}_l$  preserves the canonical class  $k_{V_l}$  of  $V_l$  up to sign and the exceptional classes up to permutation and signs. But since  $\alpha$  is not a multiple of  $k_{V_l}$ ,  $\phi$  cannot have this property. This gives a contradiction to the existence of  $\psi$  and proves the theorem in case (i). To prove the statement in case (ii) it is enough to observe that since, by what we have proved

already, the interiors of  $M(2, l, m) \# \overline{\mathbf{CP}}^2$  and  $M(l, 2, m) \# \overline{\mathbf{CP}}^2$  are not diffeomorphic, the interiors of  $M(2, l, m)$  and  $M(l, 2, m)$  cannot be diffeomorphic as well. QED

**THEOREM 2.5.** *Let  $l, m$  be positive integers with  $l > 1$ , and  $s$  a non-negative integer. If either (i)  $s \neq 0$  or (ii)  $s = 0$  and  $l \equiv 1 \pmod{2}$ , then the manifolds  $\bar{M}(1, l, m)$  and  $\bar{M}(l, 1, m)$  are homeomorphic but their interiors are not diffeomorphic.*

*Proof.* Use a sphere of square  $-1$  inside  $\bar{M}(1, l, m)$  to get a contradiction to the existence of a diffeomorphism exactly as in the proof of theorem 2.4. QED

*Proof of theorem 1.1.* Put theorems 2.4 and 2.5 together. QED

Now we give our proof of theorem 1.2.

*Proof of theorem 1.2.* In [A] it is proved that  $-W_1$  and  $-W_2$  are homeomorphic. To prove that they are not diffeomorphic, notice first that  $W_1$  can be embedded inside  $\hat{V}_k = V_k \# 2\overline{\mathbf{CP}}^2$  for any  $k \geq 1$ : blow-up two  $-1$ 's around the 0-framed knot in a link picture for  $N_k$ . This embedding sends a generator of  $H_2(W_1)$  to  $f + e \in H_2(\hat{V}_k)$ , where  $f$  is the class of a smooth fiber, and  $e$  is the Poincaré dual of an exceptional class. Next observe that a generator of  $H_2(W_2)$  is representable by a sphere  $S^2$  smoothly embedded in  $\text{int}(W_2)$  with self-intersection  $-1$ : the knot  $K_2$  is ribbon, hence slice, so take as  $S^2$  a slicing disk union the core of the corresponding 2-handle. Finally suppose, arguing by contradiction, that  $\psi: \text{int}(W_2) \cong \text{int}(W_1)$  is a diffeomorphism. The class  $f + e$  is therefore representable by the smoothly embedded sphere  $\psi(S^2)$  inside  $\hat{V}_k$ . Since  $\hat{V}_k$  does not contain embedded spheres with positive self-intersection [FM2], we may assume that  $\psi$  is orientation preserving and  $\psi(S^2)$  has self-intersection  $-2$ . Again by [FM2],  $V_k$  has big diffeomorphism group with respect to the canonical class, and it has some non-zero generalized Donaldson invariant. Moreover  $b_2^+(\hat{V}_k) \geq 3$  for  $k \geq 2$ . So when  $k \geq 2$  we may apply theorem 3.5 from [FM2], which implies that any class of square  $-1$  in  $H_2(\hat{V}_k)$  representable by a smoothly embedded 2-sphere is equal, up to sign, to the Poincaré dual of one of the exceptional classes. This is clearly not the case for  $f + e$ , so we get a contradiction to the existence of  $\psi$ . QED

## REFERENCES

- [A] AKBULUT, S. An exotic 4-manifold. *J. Diff. Geom.* 33 (1991), 357-361.
- [AK] AKBULUT, S. and R. KIRBY. Mazur manifolds. *Mich. Math. Jour.* 26 (1979), 259-284.
- [F] FREEDMAN, M.H. The topology of four-dimensional manifolds. *J. Diff. Geom.* 17 (1982), 357-453.
- [FM1] FRIEDMAN, R. and J.W. MORGAN. On the diffeomorphism types of certain algebraic surfaces. I. *J. Diff. Geom.* 27 (1988), 297-369.
- [FM2] ——— *Smooth four-manifolds and complex surfaces*. Springer-Verlag (to appear).
- [G] GOMPF, R.E. Nuclei of elliptic surfaces. *Topology* 30 (1991), 479-511.
- [K] KIRBY, R. A calculus for framed links in  $S^3$ . *Inv. Math.* 45 (1978), 36-56.

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