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It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series  $h(z)$  it will also hold for  $h(z^k)$ , at the cost of replacing  $K$  by  $K^{\frac{1}{2k}}$ . By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1 + z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10].  $\square$

COROLLARY OF PROOF. *If  $G$  satisfies the hypotheses of Theorem 2.1 (2) then for some  $k \in \mathbf{N}$ ,*

$$G(z) \underset{c}{\geq} \prod_{i=1}^{\infty} [1 + (z^k)^i]. \quad \square$$

### 3. ELLIPTIC SPACES

In this section we establish the ellipticity of the spaces listed in the introduction.

#### 3.1. Finite simply connected $H$ -spaces, $X$ .

Because  $X$  is an  $H$ -space,  $H_*(\Omega X; \mathbf{F}_p)$  is commutative, all  $p$ . Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence  $X$  is elliptic.

#### 3.2. Simply connected homogeneous spaces, $G // H$ .

We may suppose that  $G$  is simply connected, and hence elliptic by §3. The fibration  $G \rightarrow G/H \rightarrow BH$  loops to the fibration  $\Omega G \rightarrow \Omega(G/H) \rightarrow H$  in which  $\pi_1(H)$  acts trivially in  $H_*(\Omega G; \mathbf{F}_p)$  [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for  $H_*(\Omega(G/H); \mathbf{F}_p)$  from the same property for  $H_*(\Omega G; \mathbf{F}_p)$ .

#### 3.3. Fibrations $F \rightarrow X \rightarrow B$ with $F, B$ elliptic.

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that  $H_*(X; \mathbf{Z})$  is concentrated in finitely many degrees, and finitely generated in each. Hence  $X$  has the weak homotopy type of a finite  $CW$  complex. Loop the fibration  $F \rightarrow X \rightarrow B$  and use the fact that  $H_*(\Omega F; \mathbf{F}_p)$  and  $H_*(\Omega B; \mathbf{F}_p)$  grow polynomially to deduce the same property for  $H_*(\Omega X; \mathbf{F}_p)$ .

3.4. *Simply connected Poincaré complexes  $X$  with  $H^*(X; \mathbf{F}_p)$  at most doubly generated.*

Suppose  $p \neq 2$  and  $H = H^*(X; \mathbf{F}_p)$  contains an element of odd degree. Then it has an odd generator  $\alpha$ . Using Poincaré duality it is easy to see that there are only three possibilities for the algebra  $H$ :

$$H = \Lambda\alpha \quad \text{or} \quad \Lambda\alpha \otimes \Lambda\beta \quad \text{or} \quad \Lambda\alpha \otimes \mathbf{F}_p[\beta]/\beta^k.$$

In each case a simple, classical computation [11] produces  $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$  and shows that it grows polynomially. Since the Eilenberg-Moore spectral sequence converges from  $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$  to  $H^*(\Omega X; \mathbf{F}_p)$ ,  $H^*(\Omega X; \mathbf{F}_p)$  also has this property.

In all other cases ( $p = 2$  or  $H$  concentrated in even degrees)  $H$  is a commutative local ring in the classic sense. Because  $H$  satisfies Poincaré duality it is a Gorenstein ring. Now a theorem of Wiebe [12; Korollar p. 268] asserts (because  $H$  has at most two generators) that  $H$  is a polynomial algebra divided by a regular sequence. It is thus easy (and classical [11]) to compute  $\text{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ , and deduce that it grows polynomially. Hence so does  $H_*(\Omega X; \mathbf{F}_p)$ .

3.5. *Simply connected Dupin hypersurfaces  $E$  in  $S^{n+1}$ .*

In [9; Table 2.1] are listed the possibilities for  $H_*(E; \mathbf{Z})$ . We divide these into three cases, using the notation of [9].

*Case (a):  $E$  has the same integral homology as  $S^k$  or as  $S^k \times S^l$ .*

In this case Poincaré duality shows that  $E$  has the same integral cohomology ring as  $S^k$  or as  $S^k \times S^l$ , and we can apply 3.4.

*Case (b):  $E$  has the rational homotopy type of  $A_3(2)$ ,  $A_3(4)$ ,  $A_3(8)$ ,  $A_4(2)$  or  $A_6(2)$ .*

In these cases the calculations of [9; §6] show explicitly that the ring  $H^*(E; \mathbf{Z})$  is torsion free and generated by two elements. Thus each  $H^*(E; \mathbf{F}_p)$  is doubly generated, and we can apply Wiebe's result as in 3.4.

*Case (c):  $E$  has the integral homology of  $S^k \times S^l \times S^{k+l}$ , with  $k < l$ .*

We need, in this case, to recall from [9; §2] that there are linear sphere bundles

$$S^k \rightarrow E \xrightarrow{\pi_0} B \quad \text{and} \quad S^l \rightarrow E \xrightarrow{\pi_1} B_1$$

with  $B_0, B_1$  simply connected focal submanifolds of  $S^{n+1}$ . Moreover if  $D_0, D_1$  denote the corresponding disk bundles with boundary  $E$  then  $S^{n+1} = D_0 \cup_E D_1$ .

Fix  $p \geq 0$  and consider the Serre spectral sequence for the fibration  $S^k \rightarrow E \rightarrow B_0$  with coefficients in  $\mathbf{F}_p$ . If this fails to collapse then  $H^k(\pi_0): H^k(B_0; \mathbf{F}_p) \rightarrow H^k(E; \mathbf{F}_p)$  is surjective. Since  $l > k$  it is always true that  $H^k(\pi_1)$  is surjective. Choose classes  $\alpha \in H^k(B_0; \mathbf{F}_p)$ ,  $\beta \in H^k(B_1; \mathbf{F}_p)$  mapping to the same non-zero class in  $H^k(E; \mathbf{F}_p)$ . The Mayer-Vietoris sequence for the decomposition  $S^{n+1} = D_0 \cup_E D_1$  then gives a class  $\gamma \in H^k(S^{n+1}; \mathbf{F}_p)$  restricting to  $\alpha$  and  $\beta$ , which is absurd.

Thus the spectral sequence for  $S^k \rightarrow E \rightarrow B_0$  collapses and so  $H_*(B_0; \mathbf{F}_p) \cong H_*(S^l \times S^{l+k}; \mathbf{F}_p)$ . Using Poincaré duality for  $B_0$  we see that  $H^*(B_0; \mathbf{F}_p)$  and  $H^*(S^l \times S^{l+k}; \mathbf{F}_p)$  are isomorphic as graded algebras. Thus  $B_0$  is elliptic by 3.4 and  $E$  is elliptic by 3.3.

3.6. *Simply connected closed manifolds  $M$  with a smooth action by a compact Lie group  $G$ , having a simply connected codimension one orbit.*

Here we may assume  $G$  is connected. Let the orbit be  $G/K$ , and convert the inclusion of  $G/K$  into a fibration  $F \rightarrow G/K \rightarrow M$ . From [9; Table 1.5] we see that for any  $p$ ,  $\dim H_i(F; \mathbf{F}_p) \leq 2$ , all  $i$ . Thus applying the Serre spectral sequence to the fibration  $\Omega(G/K) \rightarrow \Omega M \rightarrow F$  and using 3.1 for  $G/K$  we see that  $H_*(\Omega M; \mathbf{F}_p)$  grows polynomially.

3.7. *Simply connected manifolds  $M \# N$  with each of the rings  $H^*(M; \mathbf{Z})$ ,  $H^*(N; \mathbf{Z})$  generated by a single class.*

By Van Kampen's theorem both  $M$  and  $N$  are simply connected, and so their fundamental cohomology classes are not torsion. Since each ring is monogenic,  $H^*(M; \mathbf{Z})$  and  $H^*(N; \mathbf{Z})$  are torsion free. Thus  $H^*(M; \mathbf{F}_p)$  and  $H^*(N; \mathbf{F}_p)$  are also monogenic, and so  $H^*(M \# N; \mathbf{F}_p)$  is doubly generated. Now apply 3.4.

## REFERENCES

- [1] BROWDER, W. On differential Hopf algebras. *Trans. Amer. Math. Soc.* 107 (1963), 153-176.
- [2] FELIX, Y., S. HALPERIN and J.-C. THOMAS. The homotopy Lie algebra for finite complexes. *Publ. de l'Institut des Hautes Etudes Scientifiques* 56 (1983), 387-410.
- [3] FELIX, Y., S. HALPERIN, J.-M. LEMAIRE and J.-C. THOMAS. Mod  $p$  loop space homology. *Invent. Math.* 95 (1989), 247-262.
- [4] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Hopf algebras of polynomial growth. *J. of Algebra* 125 (1989), 408-417.
- [5] FELIX, Y., S. HALPERIN and J.-C. THOMAS. Loop space homology of spaces of LS category one and two. *Math. Ann.* 287 (1990), 377-387.