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or else

(ii) There are constants K > 1 and  $N \in \mathbb{N}$  such that

$$\sum_{i=0}^{n} \dim H_{i}(\Omega X; \mathbf{F}_{p}) \geq K^{\sqrt{n}}, \quad n \geq N.$$

In case (i) the loop space homology grows at most polynomially, and X is  $\mathbf{Z}_{(p)}$ -elliptic in the sense of [6]. If (i) holds for all p then X is elliptic. The main theorems of [6] assert that if X is elliptic then X is a Poincaré complex and that  $H_*(\Omega X; \mathbf{Z})$  is a finitely generated left noetherian ring.

In case (ii) above the loop space homology grows at least semiexponentially. However, when p = 0 [2] or  $p \ge \dim X$  [8], it can be shown that even the primitive subspace of  $H_*(\Omega X; \mathbf{F}_p)$  grows exponentially (implying the same result for  $H_*(\Omega X; \mathbf{F}_p)$ ), and we conjecture that this should hold true for all p.

In the dichotomy of Theorem A, the generic situation is (ii): elliptic spaces are rare within the class of all simply connected finite CW complexes. However a number of geometrically interesting spaces are elliptic, and our second objective in this note is to show that these include the following classes of spaces (provided they are simply connected):

finite H-spaces,

homogeneous spaces,

spaces admitting a fibration  $F \rightarrow X \rightarrow B$  with F, B elliptic,

Poincaré complexes X such that for each p, the algebra  $H^*(X; \mathbf{F}_p)$  is generated by two elements,

Dupin hypersurfaces in  $S^{n+1}$ ,

closed manifolds admitting a smooth action by a compact Lie group, with a simply connected codimension one orbit,

connected sums M # N with the algebras  $H^*(M; \mathbb{Z})$  and  $H^*(N; \mathbb{Z})$  each generated by a single class.

This note is sequel to "Elliptic Spaces" [6]. In particular, it supersedes the preprint "Dupin hypersurfaces are elliptic" referred to in [6].

2. The dichotomy

Consider first any simply connected space X with each  $H_i(X; \mathbf{F}_p)$  finite dimensional. Then  $G = H_*(\Omega X; \mathbf{F}_p)$  is a graded cocommutative Hopf algebra satisfying  $G_0 = \mathbf{F}_p$  and each  $G_i$  is finite dimensional. The *depth* of G

is the least integer *m* such that  $\operatorname{Ext}_{G}^{m}(\mathbf{F}_{p}; G) \neq 0$ ; if  $\operatorname{Ext}_{G}(\mathbf{F}_{p}; G) \equiv 0$  we say *G* has *infinite depth*. In [3: Theorem A] it is shown that

depth 
$$H_*(\Omega X; \mathbf{F}_p) \leq LS \operatorname{cat} X$$
.

Thus the depth is finite when X has the weak homotopy type of a finite CW complex.

On the other hand suppose G is any graded cocommutative Hopf algebra with  $G_0 = \mathbf{F}_p$  and each  $G_i$  finite dimensional. We call G *elliptic* [7] if G is a finitely generated nilpotent Hopf algebra. According to [4; Theorem A] this is equivalent to the condition:

depth 
$$G < \infty$$
 and  $\sum_{i=0}^{n} \dim G_i \leq Cn^r (fixed C, r, all n)$ .

In view of these remarks, Theorem A follows from

THEOREM 2.1. Let G be a cocommutative Hopf algebra of finite depth such that  $G_0 = \mathbf{F}_p$  and each  $G_i$  is finite dimensional. Then there are exactly two possibilities:

(1) G is elliptic, and for some  $r \in \mathbb{N}$  there are positive constants  $C_1, C_2$  such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r, \quad n \geq 1;$$

(2) For some constants  $K > 1, N \in \mathbb{N}$ 

$$\sum_{i=0}^{n} \dim G_i \geqslant K^{\sqrt{n}}, \quad n \geqslant N.$$

*Proof.* Consider the formal power series  $G(z) = \sum_{i=0}^{\infty} \dim G_i z^i$ , and for

two formal power series  $f = \sum_{i=0}^{\infty} a_i z^i$  and  $g = \sum_{i=0}^{\infty} b_i z^i$  write  $f \leq g$  if

(2.1) 
$$\sum_{i=0}^{n} a_i \leq \sum_{i=0}^{n} b_i, \quad \text{all } n.$$

We shall first show that there are exactly two possibilities:

(2.2) For some  $r \in \mathbb{N}$  there are positive constants  $C_1, C_2$  such that

$$C_1 n^r \leq \sum_{i=0}^n \dim G_i \leq C_2 n^r$$
,  $n \geq 1$ ;

(2.3) For some  $k \in \mathbb{N}$ .

$$G(z) \geq_{c} \prod_{i=1}^{\infty} [1+(z^{k})^{i}].$$

Indeed, suppose  $\sum_{i=0}^{n} \dim G_i \leq C_2 n^r$  for all *n*, some  $C_2$  and *r*. Then by [4; Theorem B], G is elliptic and hence [7; Prop. 3.6] the formal power series G(z) has the form

$$G(z) = \frac{\prod_{j=1}^{s} (1 + z^{k_j} + \cdots + z^{(n_j-1)k_j})}{\prod_{i=1}^{r} (1 - z^{l_i})}$$

It follows at once that (2.2) is satisfied.

Conversely, we assume there is no C, r for which  $\sum_{i=0}^{n} \dim G_i \leq Cn^r$ ,

all *n*, and prove (2.3). Let  $x_1, x_2, ...$  be a sequence of generators of the algebra *G* with deg  $x_1 \leq \deg x_2 \leq \cdots$ . The subalgebra *G(i)* generated by  $x_1, ..., x_i$  is then a sub Hopf algebra. Now according to [4; Prop. 3.1] there is some *q* such that *G(i)* has finite depth,  $i \geq q$ . Moreover by [7; Prop. 3.5] *G(l)* is not elliptic for some  $l \geq q$ . Set H = G(l); it is a finitely generated non-elliptic Hopf algebra of finite depth, and dim  $G_i \geq \dim H_i$ .

Next, let *R* be the sum of the solvable normal sub Hopf algebras of *H*. Then [3; Theorem C] *R* is elliptic. Hence [7; Prop. 3.1] and [3; Prop. 3.1] the quotient Hopf algebra  $H /\!\!/ R$  has finite depth, but [7; Prop. 3.3]  $H /\!\!/ R$  is not elliptic. Clearly, however,  $H /\!\!/ R$  is finitely generated and has no central primitive elements. Now by [4; Prop. 3] there is an integer  $n_0$  and an infinite sequence of non zero primitive elements  $y_i \in H /\!\!/ R$  such that for all *i*, deg  $y_i \leq \deg y_{i+1} \leq \deg y_i + n_0$ . A linear embedding

$$\bigotimes_{i=1}^{\infty} \mathbf{F}_p[y_i] / y_i^2 \to H /\!\!/ R$$

is then defined by  $y_1^{\varepsilon_1} \otimes \cdots \otimes y_m^{\varepsilon_m} \to y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m}$ , and so

$$\prod_{i=1}^{\infty} (1+z^{\deg y_i}) \underset{c}{\ll} (H /\!\!/ R) (z) \underset{c}{\ll} H(z) \underset{c}{\ll} G(z) .$$

Since deg  $y_{i+1} \leq in_0 + \deg y_1$  it is sufficient to take  $k = \max(\deg y_1, n_0)$  to achieve (2.3).

It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series h(z) it will also hold for  $h(z^k)$ , at the cost of replacing K by  $K^{\frac{1}{2k}}$ . By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1+z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10].  $\Box$ 

COROLLARY OF PROOF. If G satisfies the hypotheses of Theorem 2.1 (2) then for some  $k \in \mathbb{N}$ ,

$$G(z) \geq_{c} \prod_{i=1}^{\infty} \left[1 + (z^{k})^{i}\right]. \qquad \Box$$

**3.** Elliptic spaces

In this section we establish the ellipticity of the spaces listed in the introduction.

3.1. Finite simply connected H-spaces, X.

Because X is an H-space,  $H_*(\Omega X; \mathbf{F}_p)$  is commutative, all p. Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence X is elliptic.

3.2. Simply connected homogeneous spaces, G // H.

We may suppose that G is simply connected, and hence elliptic by §3. The fibration  $G \to G/H \to BH$  loops to the fibration  $\Omega G \to \Omega(G/H) \to H$  in which  $\pi_1(H)$  acts trivially in  $H_*(\Omega G; \mathbf{F}_p)$  [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for  $H_*(\Omega(G/H); \mathbf{F}_p)$  from the same property for  $H_*(\Omega G; \mathbf{F}_p)$ .

# 3.3. Fibrations $F \rightarrow X \rightarrow B$ with F, B elliptic.

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that  $H_*(X; \mathbb{Z})$  is concentrated in finitely many degrees, and finitely generated in each. Hence X has the weak homotopy type of a finite CW complex. Loop the fibration  $F \to X \to B$  and use the fact that  $H_*(\Omega F; \mathbf{F}_p)$  and  $H_*(\Omega B; \mathbf{F}_p)$  grow polynomially to deduce the same property for  $H_*(\Omega X; \mathbf{F}_p)$ .