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if the limit exists. Then Veech showed that  $\mu_\theta(I)$  exists for all  $I \subseteq [0, 1)$  if and only if the partial quotients of  $\theta$  are bounded.

For other connections with ergodic theory, see the papers of Stewart [286]; del Junco [154]; Dani [70, 72]; and Baggett and Merrill [14, 15].

#### 14. PSEUDO-RANDOM NUMBER GENERATION

Lehmer [183] introduced the *linear congruential method* for pseudo-random number generation. Let  $X_0, m, a, c$  be given, and define

$$X_{k+1} = aX_k + c \pmod{m},$$

for  $k \geq 0$ . For this to be a good source of “random” numbers, we want the sequence  $X_k$  to be uniformly distributed, as well as the sequence of pairs  $(X_k, X_{k+1})$ , triples, etc.

A test for randomness called the *serial test* on pairs  $(X_k, X_{k+1})$  amounts to the two-dimensional version of the discrepancy mentioned above in Section 12. This turns out to be essentially the function  $\rho(\mathbf{g}, m)$  defined in Section 10. Thus linear congruential generators that pass the pairwise serial test arise from rationals  $a/m$  having small partial quotients in their continued fraction expansion. See the papers of Dieter [87, 88]; Niederreiter [219, 220, 222]; Knuth [170, Section 3.3.3]; and Borosh and Niederreiter [42].

#### 15. FORMAL LANGUAGE THEORY

Let  $w = w_0w_1w_2 \cdots$  be an infinite word over a finite alphabet. We say that the finite word  $x = x_0x_1 \cdots x_n$  is a *subword* of  $w$  if there exists  $m \geq 0$  such that  $w_{m+i} = x_i$ , for  $0 \leq i \leq n$ . We say that  $w$  is *k-th power free* if  $x^k$  is never a subword of  $w$ , for all nonempty words  $x$ . Here is a classical example: let  $s(n)$  denote the number of 1's in the binary expansion of  $n$ . Then the infinite word of Thue-Morse

$$t = t_0t_1t_2 \cdots = 0110100110010110 \cdots,$$

defined by  $t_n = s(n) \pmod{2}$ , is cube-free.

Another way to define infinite words is as the fixed point of a homomorphism on a finite alphabet. For example, the Thue-Morse word  $t$  is a fixed point of  $\varphi$ , where  $\varphi(0) = 01$  and  $\varphi(1) = 10$ .

A famous infinite word which has been extensively studied is the *Fibonacci word*

$$f = 101101011011010 \cdots ;$$

it is a fixed point of the homomorphism  $\mu$ , where  $\mu(1) = 10$  and  $\mu(0) = 1$ . For some of the properties of this word, see the survey of Berstel [28]. Karhumäki showed that  $f$  is fourth-power-free; see [155].

Now we define some special infinite words. Let  $\theta \in [0, 1)$  and define the infinite word  $w = w_1 w_2 w_3 \cdots$  as follows:

$$w_n = [(n+1)\theta] - [n\theta] .$$

If we set  $\theta = (\sqrt{5} - 1)/2$ , we get the Fibonacci word  $f$ . Recently, Mignosi [212] proved the following theorem: there exists a  $k$  such that  $w$  is  $k$ -th power-free, if and only if  $\theta$  has bounded partial quotients. (One direction of Mignosi's theorem follows easily from two different descriptions of  $w$  in terms of the continued fraction expansion for  $\theta$ ; see Markoff [205]; Stolarsky [287]; and Fraenkel, Mushkin, and Tassa [107].)

## 16. OTHER RESULTS

Let  $\theta$  be an irrational number of constant type. Let  $p_n/q_n$  denote the  $n$ -th convergent to  $\theta$ .

For  $n$  a positive integer, let  $P(n)$  denote the largest prime factor of  $n$ . Then given  $\varepsilon > 0$ , there exists a constant  $c = c(\theta; \varepsilon)$  such that the number of positive integers  $n \leq x$  with

$$P(q_n) < c \log \log q_n$$

is at most  $\varepsilon x$ . This is a result of Shorey [279].

Schmidt [269] showed that if  $f_1, f_2, \dots$  is a sequence of differentiable functions whose derivatives are continuous and vanish nowhere, then there are uncountably many numbers  $\theta$  such that all the numbers  $f_1(\theta), f_2(\theta), \dots$  have bounded partial quotients. For related results, see Davenport [74, 75] and Cassels [51].

Other topics connected with real numbers with bounded partial quotients not discussed in this survey include transcendental number theory (see Baker [17]; Flicker [106]; Bundschuh [49]; Angell [11]), Fibonacci hashing on digital computers (see Knuth [169, pp. 510-512]), dynamical systems and global analysis (see Deligne [81]; Katznelson [156]; Herman [142, 143, 144, 145, 146]; Meyer [211]; de la Llave [193, 194]; MacKay [196, 197]; MacKay, Meiss, and