

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 38 (1992)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY  
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**Kapitel:** 12. Discrepancy and Dispersion  
**DOI:** <https://doi.org/10.5169/seals-59489>

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Then Graham and van Lint [119] proved the following theorem:

$$\limsup_{n \rightarrow \infty} n\delta_\theta(n) < \infty$$

if and only if  $\theta$  is a number of constant type.

Boyd and Steele [43] introduced the function  $l_n^+(\theta)$ , the length of the longest increasing subsequence of  $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$ . They proved that

$$\liminf_{n \rightarrow \infty} \frac{l_n^+(\theta)}{\sqrt{n}} > 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{l_n^+(\theta)}{\sqrt{n}} < \infty$$

if and only if the partial quotients of  $\theta$  are bounded.

For some other results on  $\{n\theta\}$  connected with bounded partial quotients, see Ennola [100, 101]; Lesca [185]; Drobot [92]; and Strauch [288].

## 12. DISCREPANCY AND DISPERSION

Let  $\omega = (x_1, x_2, x_3, \dots)$  be a sequence of real numbers. Let  $I \subseteq [0, 1)$  be an interval and let  $|I|$  denote its length. Define the counting function  $S_n(I) = S_n(I, \omega)$  as the number of terms  $x_k, 1 \leq k \leq n$ , for which  $\{x_k\} \in I$ .

The *discrepancy*  $D_n(x_1, x_2, \dots, x_n)$  is a measure of how much the sequence  $x_1, x_2, \dots, x_n$  deviates from a uniform distribution. It is defined as follows:

$$D_n(\omega) = D_n(x_1, x_2, \dots, x_n) = \sup_{I \subseteq [0, 1)} \left| \frac{S_n(I, \omega)}{n} - |I| \right|.$$

Now consider the discrepancy of the sequence  $\omega = (\theta, 2\theta, 3\theta, \dots)$ . If  $\theta$  has bounded partial quotients, then the discrepancy of  $\omega$  is small. In particular, we have the following estimate: If  $K(\theta) \leq k$ , then

$$nD_n(\omega) \leq 3 + \left( \frac{1}{\log \alpha} + \frac{k}{\log(k+1)} \right) \log n$$

for  $\alpha = \frac{1}{2}(1 + \sqrt{5})$ . See, for example, Kuipers and Niederreiter [173].

For other results connecting discrepancy and the boundedness of the partial quotients, see the papers of Niederreiter [218] and Dupain and Sós [94, 95]. Also see Beck and Chen [25] and Richert [258].

We can also consider the so-called  $L^2$  discrepancy,  $T_n$ , defined as follows: let

$$R_n(t) = \frac{S_n([0, t), \omega)}{n} - t$$

and put

$$T_n(\omega) = \left( \int_0^1 R_n^2(t) dt \right)^{1/2}.$$

It is possible to generalize the definitions of  $D_n$  and  $T_n$  to the multi-dimensional case, though we omit the details. By appealing to numbers with bounded partial quotients, Davenport [73] constructed sequences in two dimensions with low  $L^2$  discrepancy. Also see Proinov [250, 251, 252].

Another measure connected with sequences is called *dispersion*. Let  $\omega = (x_1, x_2, \dots)$  and define the dispersion

$$d_n(\omega) = \sup_{x \in [0, 1]} \min_{1 \leq k \leq n} |x - x_k|,$$

essentially half the distance between the most widely separated points of the sequence  $x_1, x_2, \dots, x_n$ . (Compare with the function  $\delta_\theta(n)$  in Section 11.)

Niederreiter [221] considered the dispersion of the sequence  $\{n\theta\}$ . He showed that if  $\theta$  has bounded partial quotients, then  $d_n(\omega) = O(1/n)$ . He also gave a more detailed estimate, showing that  $d_n(\omega)$  is approximately  $K(\theta)/4n$ . Also see Drobot [93] and Larcher [311].

### 13. CONNECTIONS WITH ERGODIC THEORY

Let  $\theta$  be irrational,  $\omega = (\theta, 2\theta, \dots)$  and  $S_n(I, \omega)$  be defined as in the previous section. Veech [293] developed connections between  $S_n$  and ergodic theory. We mention one result that is number-theoretic in nature. Let  $x_n = S_n(I, \omega) \bmod 2$ , and define

$$\mu_\theta(I) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq k \leq n} x_k,$$