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number. It follows from the results of Kmošek and Shallit cited above that  $Z(2^{2^k-1}) \leq 2$  for all  $k \geq 0$ .

Borosh and Niederreiter [42] showed that  $Z(2^k) \leq 3$  for  $6 \leq k \leq 35$ .

More recently, Niederreiter [223] proved that Zaremba's conjecture holds for all powers of 2; in fact, we have  $Z(2^k) \leq 3$  for all  $k \geq 0$ .

Larcher [182, Corollary 2] proved the existence of a constant  $c$ , such that for every  $n \geq 1$  there exists a positive integer  $j \leq n$ , relatively prime to  $n$ , such that if

$$j/n = [0, a_1, a_2, \dots, a_m],$$

then

$$\sum_{1 \leq i \leq m} a_i < c(\log n)(\log \log n)^2.$$

This is close to the best possible bound  $O(\log n)$ , which was reportedly conjectured by L. Moser (although I do not know a reference); the bound would be a consequence of Zaremba's conjecture.

For other results connected with Zaremba's conjecture, see the papers of Cusick [63, 66]; Niederreiter [224]; Sander [268]; and Hensley [315].

## 11. PROPERTIES OF THE SEQUENCE $n\theta \pmod{1}$

If  $\theta$  is a real number, by  $\theta \pmod{1}$  we mean  $\{\theta\} = \theta - [\theta]$ , the fractional part of  $\theta$ .

It has been known at least since Bernoulli [26] that properties of the sequence  $\theta, 2\theta, 3\theta, \dots$  are intimately connected with the continued fraction expansion for  $\theta$ . The distribution of  $n\theta \pmod{1}$  is a vast subject, and we restrict ourselves to mentioning several results connected with numbers of constant type.

Let  $\theta$  be an irrational number, and let

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$$

be the sequence of points  $\{k\theta\}$ ,  $1 \leq k \leq n$ , arranged in ascending order. Define

$$\delta_\theta(n) = \max_{1 \leq i \leq n+1} a_i - a_{i-1}.$$

Then Graham and van Lint [119] proved the following theorem:

$$\limsup_{n \rightarrow \infty} n\delta_\theta(n) < \infty$$

if and only if  $\theta$  is a number of constant type.

Boyd and Steele [43] introduced the function  $l_n^+(\theta)$ , the length of the longest increasing subsequence of  $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$ . They proved that

$$\liminf_{n \rightarrow \infty} \frac{l_n^+(\theta)}{\sqrt{n}} > 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{l_n^+(\theta)}{\sqrt{n}} < \infty$$

if and only if the partial quotients of  $\theta$  are bounded.

For some other results on  $\{n\theta\}$  connected with bounded partial quotients, see Ennola [100, 101]; Lesca [185]; Drobot [92]; and Strauch [288].

## 12. DISCREPANCY AND DISPERSION

Let  $\omega = (x_1, x_2, x_3, \dots)$  be a sequence of real numbers. Let  $I \subseteq [0, 1]$  be an interval and let  $|I|$  denote its length. Define the counting function  $S_n(I) = S_n(I, \omega)$  as the number of terms  $x_k$ ,  $1 \leq k \leq n$ , for which  $\{x_k\} \in I$ .

The *discrepancy*  $D_n(x_1, x_2, \dots, x_n)$  is a measure of how much the sequence  $x_1, x_2, \dots, x_n$  deviates from a uniform distribution. It is defined as follows:

$$D_n(\omega) = D_n(x_1, x_2, \dots, x_n) = \sup_{I \subseteq [0, 1]} \left| \frac{S_n(I, \omega)}{n} - |I| \right|.$$

Now consider the discrepancy of the sequence  $\omega = (\theta, 2\theta, 3\theta, \dots)$ . If  $\theta$  has bounded partial quotients, then the discrepancy of  $\omega$  is small. In particular, we have the following estimate: If  $K(\theta) \leq k$ , then

$$nD_n(\omega) \leq 3 + \left( \frac{1}{\log \alpha} + \frac{k}{\log(k+1)} \right) \log n$$

for  $\alpha = \frac{1}{2}(1 + \sqrt{5})$ . See, for example, Kuipers and Niederreiter [173].