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Kapitel: 11. Properties of the sequence n (mod 1)

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number. It follows from the results of Kmošek and Shallit cited above that $Z(2^{2^k-1}) \leqslant 2$ for all $k \geqslant 0$.

Borosh and Niederreiter [42] showed that $Z(2^k) \leqslant 3$ for $6 \leqslant k \leqslant 35$.

More recently, Niederreiter [223] proved that Zaremba's conjecture holds for all powers of 2; in fact, we have $Z(2^k) \leqslant 3$ for all $k \geqslant 0$.

Larcher [182, Corollary 2] proved the existence of a constant c, such that for every $n \geqslant 1$ there exists a positive integer $j \leqslant n$, relatively prime to n, such that if

$$j/n = [0, a_1, a_2, \ldots, a_m] ,$$

then

$$\sum_{1 \leqslant i \leqslant m} a_i < c(\log n)(\log \log n)^2 .$$

This is close to the best possible bound $O(\log n)$, which was reportedly conjectured by L. Moser (although I do not know a reference); the bound would be a consequence of Zaremba's conjecture.

For other results connected with Zaremba's conjecture, see the papers of Cusick [63, 66]; Niederreiter [224]; Sander [268]; and Hensley [315].

11. Properties of the sequence $n\theta$ (mod 1)

If θ is a real number, by θ (mod 1) we mean $\{\theta\} = \theta - [\theta]$, the fractional part of θ.

It has been known at least since Bernoulli [26] that properties of the sequence $\theta, 2\theta, 3\theta, \ldots$ are intimately connected with the continued fraction expansion for θ. The distribution of $n\theta$ (mod 1) is a vast subject, and we restrict ourselves to mentioning several results connected with numbers of constant type.

Let θ be an irrational number, and let

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} = 1$$

be the sequence of points $\{k\theta\}$, $1 \leqslant k \leqslant n$, arranged in ascending order. Define

$$\delta_\theta(n) = \max_{1 \leqslant i \leqslant n+1} a_i - a_{i-1} .$$

Then Graham and van Lint [119] proved the following theorem:

$$\limsup_{n\to\infty} n\delta_\theta(n) < \infty$$

if and only if θ is a number of constant type.

Boyd and Steele [43] introduced the function $l_n^+(\theta)$, the length of the longest increasing subsequence of $\{\theta\}, \{2\theta\}, \ldots, \{n\theta\}$. They proved that

$$\liminf_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} > 0$$

and

$$\limsup_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} < \infty$$

if and only if the partial quotients of θ are bounded.

For some other results on $\{n\theta\}$ connected with bounded partial quotients, see Ennola [100, 101]; Lesca [185]; Drobot [92]; and Strauch [288].

12. Discrepancy and Dispersion

Let $\omega = (x_1, x_2, x_3, \ldots)$ be a sequence of real numbers. Let $I \subseteq [0, 1)$ be an interval and let $|I|$ denote its length. Define the counting function $S_n(I) = S_n(I, \omega)$ as the number of terms x_k, $1 \leqslant k \leqslant n$, for which $\{x_k\} \in I$.

The *discrepancy* $D_n(x_1, x_2, \ldots, x_n)$ is a measure of how much the sequence $x_1, x_2, \ldots, x_n$ deviates from a uniform distribution. It is defined as follows:

$$D_n(\omega) = D_n(x_1, x_2, \ldots, x_n) = \sup_{I \subseteq [0,1)} \left| \frac{S_n(I, \omega)}{n} - |I| \right| .$$

Now consider the discrepancy of the sequence $\omega = (\theta, 2\theta, 3\theta, \ldots)$. If θ has bounded partial quotients, then the discrepancy of ω is small. In particular, we have the following estimate: If $K(\theta) \leqslant k$, then

$$nD_n(\omega) \leqslant 3 + \left(\frac{1}{\log \alpha} + \frac{k}{\log(k+1)} \right) \log n$$

for $\alpha = \frac{1}{2}(1 + \sqrt{5})$. See, for example, Kuipers and Niederreiter [173].