

# 8. Hall's theorem

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Good also obtained the estimate  $.5306 < \dim \mathcal{E}_2 < .5320$ . This was improved by Bumby [48] in 1985 to  $.5312 \leq \dim \mathcal{E}_2 \leq .5314$ . More recently, Hensley [140] showed that  $.53128049 < \dim \mathcal{E}_2 < .53128051$ . For other results on the Hausdorff dimension of  $\mathcal{E}_k$  and related sets, see Jarník [153]; Besicovitch [30]; Rogers [262]; Baker and Schmidt [21]; Hirst [147, 148]; Billingsley and Henningsen [32]; Cusick [63, 64, 65]; Pollington [245]; Kaufman [158]; Marion [202]; Gardner and Mauldin [115]; Ramharter [253, 254]; and Hensley [139, 141, 308, 309].

## 7. SCHMIDT'S GAME

W. M. Schmidt [270] introduced the following two-player game, called an  $(\alpha, \beta)$  game: let  $\alpha, \beta$  be real numbers with  $0 < \alpha, \beta < 1$ . First Bob chooses a closed interval on the real line, called  $B_1$ . Then Alice chooses a closed interval  $A_1 \subset B_1$ , such that the length of  $A_1$  is  $\alpha$  times the length of  $B_1$ . Then Bob chooses a closed interval  $B_2 \subset A_1$ , such that the length of  $B_2$  is  $\beta$  times the length of  $A_1$ , and so on. If the intersection of all the intervals  $A_i$  is a number with bounded partial quotients, then Alice is declared the winner; otherwise Bob is declared the winner.

Schmidt showed that if  $0 < \alpha < 1/2$ , then Alice always has a winning strategy for this game. This is somewhat surprising, since as we have seen above, the set  $\mathcal{E}$  of numbers with bounded partial quotients has Lebesgue measure 0.

Using the theory of  $(\alpha, \beta)$  games, Schmidt also reproved the result of Jarník that  $\mathcal{E}$  has Hausdorff dimension 1.

Several papers have proved other results on  $(\alpha, \beta)$  games: see Schmidt [271]; Freiling [109, 110]; and Dani [70, 71, 72]. Also see Schmidt [272, Chapter 3].

## 8. HALL'S THEOREM

If  $S$  and  $T$  are sets, then by  $S + T$  we mean the set

$$\{s + t \mid s \in S, t \in T\}.$$

Similarly, by  $S \cdot T$  we mean the set

$$\{st \mid s \in S, t \in T\}.$$

If  $S$  is a set of Lebesgue measure zero, then it is quite possible for  $S + S$  to have positive measure. For example, if  $C$  denotes the Cantor set (numbers

in  $[0, 1]$  containing only 0's and 2's in their ternary expansion), then  $C$  has measure 0, and it is not hard to show that  $C + C = [0, 2]$ ; see Borel [40] or Pavone [233]. The result is due to Steinhaus [310]; I am most grateful to G. Myerson for bringing this to my attention.

As we have seen above, the set  $\mathcal{B}$ , and hence each  $\mathcal{B}_k$ , also has Lebesgue measure zero. In 1947 Hall proved the following theorem [126]:

**THEOREM 3.** *Every real number  $x$  can be written as  $x = y + z$ , where  $y, z \in \mathcal{B}_4$ . Every real number  $x \geq 1$  can be written as  $x = yz$ , where  $y, z \in \mathcal{B}_4$ .*

An exposition of Hall's result can be found in Cusick and Flahive [67].

Using the notation of the first paragraph of this section, we could rephrase the statement of Hall's theorem as follows:  $\mathcal{B}_4 + \mathcal{B}_4 = \mathbf{R}$ , and  $[1, \infty) \subseteq \mathcal{B}_4 \cdot \mathcal{B}_4$ .

In 1973, Cusick [61] proved that  $\mathcal{B}_3 + \mathcal{B}_3 + \mathcal{B}_3 = \mathbf{R}$ , and  $\mathcal{B}_2 + \mathcal{B}_2 + \mathcal{B}_2 = \mathbf{R}$ . He also observed that  $\mathcal{B}_3 + \mathcal{B}_3 \neq \mathbf{R}$ , and  $\mathcal{B}_2 + \mathcal{B}_2 + \mathcal{B}_2 \neq \mathbf{R}$ . These results were independently discovered by Diviš [90] and J. Hlavka<sup>1)</sup> [149]. Hlavka also showed that  $\mathcal{B}_3 + \mathcal{B}_4 = \mathbf{R}$ , and similar results. Apparently the status of  $\mathcal{B}_2 + \mathcal{B}_5$  and  $\mathcal{B}_2 + \mathcal{B}_6$  is still open.

For results of a similar character, see Cusick [60]; Cusick and Lee [68]; and Bumby [47].

## 9. EXPLICIT EXAMPLES OF TRANSCENDENTAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS

In Lang [179] we find the following statement:

No simple example of [irrational] numbers of constant type, other than the one given above [real quadratic irrationals], is known. The best guess is that there are no other "natural" examples.

(Also see Lang [180].)

However, in 1979 Kmošek [167] and Shallit [275] independently discovered the following "natural" example of numbers of constant type.

**THEOREM 4.** *Let  $n \geq 2$  be an integer and define*

$$(I) \quad f(n) = \sum_{i \geq 0} n^{-2^i}.$$

<sup>1)</sup> Note this is *not* same person as E. Hlawka!