

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 38 (1992)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY
Autor: Shallit, Jeffrey
Kapitel: 6. Fractal Geometry
DOI: <https://doi.org/10.5169/seals-59489>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

They showed that if θ has bounded partial quotients, then $U_n(\theta) = O(\log n)$, $V_n(\theta) = O(n)$, and $W_n(\theta) = O(n^2)$. See [134].

(Warning to the reader: in their papers, Hardy and Littlewood used the notation $\{x\}$ to mean $x - [x] - \frac{1}{2}$, not $x - [x]$, as is more standard today.)

For other related papers, see Hardy and Littlewood [129, 131]; the collected works of Hardy [127]; Ostrowski [230]; Khintchine [161]; Oppenheim [228]; Chowla [53, 54]; Walfisz [298, 299, 300], and Schoissengeier [314].

Others researchers have examined similar sums in connection with numbers with bounded partial quotients. See the papers of Faĭziev [104] Ivanov [151], and Schoissengeier [274].

6. FRACTAL GEOMETRY

Numbers with bounded partial quotients provided an early example of a set with non-integral Hausdorff dimension.

Let $\dim S$ denote the Hausdorff dimension of the set S (for a definition, see, e.g. Falconer [103]). We use the definitions of \mathcal{E} and \mathcal{E}_k from section 1.

In 1928, Jarník [152] proved that $\dim \mathcal{E} = 1$,

$$\frac{1}{4} < \dim \mathcal{E}_2 < 1,$$

and

$$1 - \frac{4}{k \log 2} < \dim \mathcal{E}_k < 1 - \frac{1}{8k \log k},$$

for $k > 8$. An exposition of Jarník's work can be found in Rogers [263].

In 1941, Good proved the following result [118]:

$$\dim \mathcal{E}_k = \lim_{n \rightarrow \infty} \sigma_{k,n},$$

where $\sigma = \sigma_{k,n}$ is the real root of the equation

$$\sum_{1 \leq a_1, a_2, \dots, a_n \leq k} Q(a_1, a_2, \dots, a_n)^{-2\sigma} = 1$$

and $Q(\)$ denotes Euler's continuant polynomial. (These are multivariate polynomials, defined by $Q(\) = 1$, $Q(a_1) = a_1$, and

$$Q(a_1, a_2, \dots, a_n) = a_n Q(a_1, a_2, \dots, a_{n-1}) + Q(a_1, a_2, \dots, a_{n-2})$$

for $n \geq 2$.)

Good also obtained the estimate $.5306 < \dim \mathcal{E}_2 < .5320$. This was improved by Bumby [48] in 1985 to $.5312 \leq \dim \mathcal{E}_2 \leq .5314$. More recently, Hensley [140] showed that $.53128049 < \dim \mathcal{E}_2 < .53128051$. For other results on the Hausdorff dimension of \mathcal{E}_k and related sets, see Jarník [153]; Besicovitch [30]; Rogers [262]; Baker and Schmidt [21]; Hirst [147, 148]; Billingsley and Henningsen [32]; Cusick [63, 64, 65]; Pollington [245]; Kaufman [158]; Marion [202]; Gardner and Mauldin [115]; Ramharter [253, 254]; and Hensley [139, 141, 308, 309].

7. SCHMIDT'S GAME

W. M. Schmidt [270] introduced the following two-player game, called an (α, β) game: let α, β be real numbers with $0 < \alpha, \beta < 1$. First Bob chooses a closed interval on the real line, called B_1 . Then Alice chooses a closed interval $A_1 \subset B_1$, such that the length of A_1 is α times the length of B_1 . Then Bob chooses a closed interval $B_2 \subset A_1$, such that the length of B_2 is β times the length of A_1 , and so on. If the intersection of all the intervals A_i is a number with bounded partial quotients, then Alice is declared the winner; otherwise Bob is declared the winner.

Schmidt showed that if $0 < \alpha < 1/2$, then Alice always has a winning strategy for this game. This is somewhat surprising, since as we have seen above, the set \mathcal{E} of numbers with bounded partial quotients has Lebesgue measure 0.

Using the theory of (α, β) games, Schmidt also reproved the result of Jarník that \mathcal{E} has Hausdorff dimension 1.

Several papers have proved other results on (α, β) games: see Schmidt [271]; Freiling [109, 110]; and Dani [70, 71, 72]. Also see Schmidt [272, Chapter 3].

8. HALL'S THEOREM

If S and T are sets, then by $S + T$ we mean the set

$$\{s + t \mid s \in S, t \in T\}.$$

Similarly, by $S \cdot T$ we mean the set

$$\{st \mid s \in S, t \in T\}.$$

If S is a set of Lebesgue measure zero, then it is quite possible for $S + S$ to have positive measure. For example, if C denotes the Cantor set (numbers