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<b>Autor:</b>	Shallit, Jeffrey
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## 5. CERTAIN SUMS IN DIOPHANTINE APPROXIMATION

Let us agree to write  $\{\theta\}$  for the fractional part of  $\theta$ , namely,  $\theta - [\theta]$ .

One of the earliest appearances of real numbers with bounded partial quotients is in the theory of Diophantine approximation.

For example, consider the sum

$$s'_n(\theta) = \sum_{1 \leq k \leq n} \left( \{k\theta\} - \frac{1}{2} \right).$$

Clearly  $s'_n(\theta) = O(n)$ ; but Lerch proved in 1904 that if  $\theta$  has bounded partial quotients, we have  $s'_n(\theta) = O(\log n)$ . See [184]. (This result was also announced by Hardy and Littlewood in 1912; see [128].)

At the International Congress of Mathematicians in 1912, Hardy and Littlewood [128] announced several theorems on Diophantine approximation, some of which relate to the subject at hand. For example, they defined

$$s_n(\theta) = \sum_{1 \leq k \leq n} e^{\left(k - \frac{1}{2}\right)^2 \pi i \theta},$$

and stated that if  $\theta$  has bounded partial quotients, then  $s_n(\theta) = O(\sqrt{n})$ . The proof appeared later; see [130].

At the same Congress, Hardy and Littlewood announced that

$$\sum_{1 \leq k \leq n} \left( \{k\theta\} - \frac{1}{2} \right)^2 = \frac{n}{12} + O(1),$$

for all irrational  $\theta$ . This is incorrect, and the correct formulation was stated in a 1922 paper: the result holds for many, but not all irrationals, and in particular it holds for  $\theta$  with bounded partial quotients. See [132] for the statement and [133] for a proof.

Hardy and Littlewood also examined other series of interest. They defined:

$$U_n(\theta) = \sum_{1 \leq k \leq n} \frac{(-1)^k}{k \sin k\pi\theta},$$

$$V_n(\theta) = \sum_{1 \leq k \leq n} \frac{(-1)^k}{\sin k\pi\theta},$$

and

$$W_n(\theta) = \sum_{1 \leq k \leq n} \frac{1}{(\sin k\pi\theta)^2}.$$

They showed that if  $\theta$  has bounded partial quotients, then  $U_n(\theta) = O(\log n)$ ,  $V_n(\theta) = O(n)$ , and  $W_n(\theta) = O(n^2)$ . See [134].

(Warning to the reader: in their papers, Hardy and Littlewood used the notation  $\{x\}$  to mean  $x - [x] - \frac{1}{2}$ , not  $x - [x]$ , as is more standard today.)

For other related papers, see Hardy and Littlewood [129, 131]; the collected works of Hardy [127]; Ostrowski [230]; Khintchine [161]; Oppenheim [228]; Chowla [53, 54]; Walfisz [298, 299, 300], and Schoissengeier [314].

Others researchers have examined similar sums in connection with numbers with bounded partial quotients. See the papers of Faĭziev [104] Ivanov [151], and Schoissengeier [274].

## 6. FRACTAL GEOMETRY

Numbers with bounded partial quotients provided an early example of a set with non-integral Hausdorff dimension.

Let  $\dim S$  denote the Hausdorff dimension of the set  $S$  (for a definition, see, e.g. Falconer [103]). We use the definitions of  $\mathcal{E}$  and  $\mathcal{E}_k$  from section 1.

In 1928, Jarník [152] proved that  $\dim \mathcal{E} = 1$ ,

$$\frac{1}{4} < \dim \mathcal{E}_2 < 1,$$

and

$$1 - \frac{4}{k \log 2} < \dim \mathcal{E}_k < 1 - \frac{1}{8k \log k},$$

for  $k > 8$ . An exposition of Jarník's work can be found in Rogers [263].

In 1941, Good proved the following result [118]:

$$\dim \mathcal{E}_k = \lim_{n \rightarrow \infty} \sigma_{k,n},$$

where  $\sigma = \sigma_{k,n}$  is the real root of the equation

$$\sum_{1 \leq a_1, a_2, \dots, a_n \leq k} Q(a_1, a_2, \dots, a_n)^{-2\sigma} = 1$$

and  $Q()$  denotes Euler's continuant polynomial. (These are multivariate polynomials, defined by  $Q() = 1$ ,  $Q(a_1) = a_1$ , and

$$Q(a_1, a_2, \dots, a_n) = a_n Q(a_1, a_2, \dots, a_{n-1}) + Q(a_1, a_2, \dots, a_{n-2})$$

for  $n \geq 2$ .)