

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	38 (1992)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY
<b>Autor:</b>	Shallit, Jeffrey
<b>Kapitel:</b>	1. Introduction and Definitions
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-59489">https://doi.org/10.5169/seals-59489</a>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY

by Jeffrey SHALLIT

**ABSTRACT.** Real numbers with bounded partial quotients in their continued fraction expansion appear in many different fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications.

### 1. INTRODUCTION AND DEFINITIONS

If  $x$  is a real number, we can expand  $x$  as a *simple continued fraction*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which we abbreviate in this paper as

$$x = [a_0, a_1, a_2, a_3, \dots].$$

In this paper, we only discuss the case of *regular continued fractions*, where  $a_0$  is an integer and  $a_i$ , is a positive integer for  $i \geq 1$ ; the expansion may or may not terminate. (For an introduction to continued fractions, see Hardy and Wright [135, Chap. 10]; for a more definitive work, see Perron [236]. For a history of continued fractions, see Brezinski [44].)

---

Supported in part by NSF Grant CCS-8817400, a Walter Burke Award from Dartmouth College and NSERC Canada.

If  $x$  is rational, then its continued fraction expansion terminates, and we can write  $x = [a_0, a_1, \dots, a_n]$ . If we agree that  $a_n = 1$  and  $n \geq 1$ , then this expansion is unique and we define

$$K(x) = \max_{1 \leq k \leq n} a_k,$$

the largest partial quotient in the continued fraction for  $x$ .

If  $x$  is irrational, then its continued fraction expansion does not terminate. This expansion is unique. We write  $x = [a_0, a_1, a_2, \dots]$  and define

$$K(x) = \sup_{k \geq 1} a_k.$$

If  $K(x) < \infty$ , then we say that  $x$  has *bounded partial quotients*.

We define  $\mathcal{B}_k = \{x \in \mathbf{R} \mid K(x) \leq k\}$ , and  $\mathcal{B} = \{x \in \mathbf{R} \mid K(x) < \infty\}$ . Furthermore, let  $\mathcal{E}_k = \mathcal{B}_k \cap (0, 1)$  and  $\mathcal{E} = \mathcal{B} \cap (0, 1)$ .

Real numbers with bounded partial quotients appear in many fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications. Because of limited space, we cannot include a discussion of every result in detail. However, we have tried to include as complete a list of references as possible for those topics directly related to the main subject. Readers who know of other references are urged to contact the author (and provide a copy of the relevant paper, if possible). It is hoped that the list of references may contain some surprises even for experts in the field.

The author's interest in the subject arose from the material in Section 9. Because of this, the viewpoint presented in this article may be somewhat idiosyncratic.

## 2. NUMBERS OF CONSTANT TYPE

Let  $\theta$  be an irrational number, and let  $\|\theta\|$  denote the distance between  $\theta$  and the closest integer.

Let  $r \geq 1$  be a real number. We say that  $\theta$  is *of type*  $< r$  if

$$q \|\theta\| \geq \frac{1}{r}$$

for all integers  $q \geq 0$ . Then we have the following