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AN ANALOGUE OF HUBER'S FORMULA FOR RIEMANN'S ZETA FUNCTION

by Floyd L. WILLIAMS [1])

To the memory of Michio Kuga

1. INTRODUCTION

A remarkable formula of H. Huber [11] relates the class 1 spectrum of a compact Riemann surface X and the spectral zeros of Selberg's zeta function ζ_X of X. More generally, if X is a space form (not necessarily compact) of a rank 1 symmetric space one can still assign to X a Selberg zeta function ζ_X and formulate a generalized version of Huber's formula [6], [18]. Here a decisive role is played by the Selberg trace formula.

On the other hand Weil's explicit sum formula for Riemann's zeta function ζ [16], [17] bears some striking similarity in appearance to the trace formula. It is now known, as a matter of fact, by a recent work of D. Goldfeld [7] that there exists a kernel function on a suitable space such that the conjugacy class sum in Selberg's trace formula is precisely the sum over the primes of Weil's formula; i.e. Weil's formula indeed can be interpreted as a trace formula.

Motivated by a certain "radial" function which occurs in semisimple Lie theory we consider a certain test function which we plug into Weil's formula, and we derive thereby a formula of Huber type for ζ. The formula, see Theorem 7.1, involves a sum over the "spectrum" of ζ — i.e. over its non-trivial zeros. We derive a second interesting formula in Theorem 7.10 by specializing the parameter s in Theorem 7.1.

Although Lie theory and spectral theory serve as a context and motivation, the lecture which is largely self-contained requires no familiarity with these subjects. We assume only a knowledge of basic real and complex analysis.

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[1]) This is an expanded version of an invited Mathematical Association of America address delivered at the winter San Francisco meeting on January 16, 1991.

The author is thankful for and honored by the invitation extended by the Program Committee of the Mathematical Association of America to deliver this lecture. We dedicate the lecture to the memory of a very great mathematician — a kind and humble man — a friend — Professor Michio Kuga.

2. Outline of the Lecture

I. Huber's formula (as a context)
II. Riemann's zeta function — basic facts and the Riemann hypothesis (= RH)
III. Test functions
IV. Weil's explicit formula
V. The Schwartz space and the RH
VI. The main test function
VII. An analogue of Huber's formula

I. Huber's formula (as a context)

Since Huber's formula provides the motivation for this lecture we shall state (for the record) this remarkable result. Neither the result nor any understanding of it is required for later purposes.

Let G denote the group $SL(2, \mathbf{R})$ of real 2×2 matrices with determinant equal to 1:

$$G = SL(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} a, b, c, d, \in \mathbf{R} \\ ad - cb = 1 \end{matrix} \right\} \tag{1.1}$$

where $\mathbf{R}$ denotes the field of real numbers. Let $\Gamma \subset G$ be a discrete torsionfree[1]) subgroup such that the quotient $\Gamma \backslash G$ is compact. Euler's classical gamma function will also be denoted by Γ:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt\,, \qquad \operatorname{Re} s > 0\,. \tag{1.2}$$

The function $N: \Gamma - \{1\} \to \mathbf{R}$ defined by

$$N(P) = \max |c|^2\,, \qquad c = \text{an eigenvalue of } P \tag{1.3}$$

[1]) That is, if $\gamma^n = 1$ for $\gamma \in \Gamma$, $n > 0$ an integer, then $\gamma = 1$.