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REMARK ON THE UNIQUENESS OF THE FINITE PACKINGS  
OF MAXIMAL DENSITY

All natural or practical examples of circle packings such as bees living in a honeycomb or a bundle of fibre-glass optical tubes are always packing problems of a finite number of circles (i.e. packings of their cross-section circles). The infinite circle packings of the entire plane of  $E^2$  are actually the limit situation of the finite circle packings. Therefore, it is natural to give an appropriate definition of the concept of global density for a finite circle packing. We propose the following definition of a cluster of circles and the (global) density of a cluster of circles, namely

*Definition.* A packing of finite number of equal circles is called a *cluster of circles* if any two of them can be linked through neighboring pairs of center distances less than  $2\sqrt{2}$  times the radii.

Let  $\mathcal{C}$  be a given cluster of circles. Then, an extension,  $\mathcal{C}^*$ , of  $\mathcal{C}$  is called a *saturated coating of  $\mathcal{C}$*  if all circles of  $\mathcal{C}^* \setminus \mathcal{C}$  are neighbors of some circles in  $\mathcal{C}$  and it is impossible to add any more such neighbors to  $\mathcal{C}^*$ . Observe that *every circle in  $\mathcal{C}$  has a saturated set of neighbors in  $\mathcal{C}^*$*  and hence has a well-defined *local cell* with respect to  $\mathcal{C}^*$ . The *usual weighted average* of all the local densities of circles in  $\mathcal{C}$  with respect to the given saturated coating  $\mathcal{C}^*$  is defined to be the density of  $\mathcal{C}$  in  $\mathcal{C}^*$ , i.e.  $\rho(\mathcal{C} \text{ rel } \mathcal{C}^*)$ .

*Definition:* The global density of  $\mathcal{C}$  is defined to be the least upper bound of the densities of  $\mathcal{C}$  in all possible saturated coatings of  $\mathcal{C}$ , namely

$$\rho(\mathcal{C}) = \text{l.u.b.} \{ \rho(\mathcal{C} \text{ rel } \mathcal{C}^*) \}$$

where  $\mathcal{C}^*$  run through all possible saturated coatings of  $\mathcal{C}$ .

**UNIQUENESS THEOREM** (On finite circle packings of maximal density).  $\pi/\sqrt{12}$  is still the maximal possible global density of all clusters of circles, and the global density of a cluster of circles,  $\mathcal{C}$ , attains the above maximum of  $\pi/\sqrt{12}$  when and only when  $\mathcal{C}$  is a subcluster of circles in the hexagon packing.

*Proof.* It is again a direct consequence of the above Theorem on the maximal local density and its uniqueness.