# 2. Complex growth series

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Thus the equality (1.1) holds in that case.

2) Assume

$$W = < s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = 1 >$$

to be a Coxeter group of type $\tilde{A}_2$. The geometric realisation of $\Sigma(W, S)$ is a plane. We have

$$W_S(t_1, t_2) = \frac{(1 + t_1)(1 + t_1 + t_1^2)}{(1 - t_1)(1 - t_1^2)} + 3\frac{1 + t_1 + t_1^2}{(1 - t_1)(1 - t_1^2)} t_2 + 3\frac{1}{(1 - t_1)(1 - t_1^2)} t_2^2 + t_2^3,$$

and

$$W_S(t) = \frac{(1 + t)(1 + t + t^2)}{(1 - t)(1 - t^2)},$$

$$W_{\{s_1, s_2\}}(t) = W_{\{s_1, s_3\}}(t) = W_{\{s_2, s_3\}}(t) = (1 + t)(1 + t + t^2),$$

$$W_{\{s_1\}}(t) = W_{\{s_2\}}(t) = W_{\{s_3\}}(t) = 1 + t,$$

$$W_\emptyset(t) = 1.$$

Thus the equality (1.1) holds in that case.

In Section 2 we will recall some definitions in the theory of Coxeter complexes, we will define the complex growth series of a Coxeter system $(W, S)$, we will prove that $W_S(t_1, 0) = W_S(t_1)$ and $W_S(0, t_2) = (1 + t_2)^{|S|}$ (Proposition 1), we will prove the equalities (1.4) and (1.5) (Proposition 2), and we will prove the Main Theorem.

## 2. Complex growth series

We assume the reader to be familiar with the notions of simplicial complex, chamber complex, adjacency between two chambers, gallery and labelling. We refer to [2, Chap. I, Appendix] for a good exposition of these notions.

Let $(W, S)$ be a Coxeter system. A *special coset* of $(W, S)$ is a coset $wW_X$, with $w \in W$ and $X \subseteq S$. We denote by $\Sigma = \Sigma(W, S)$ the poset of all special cosets, ordered by the reverse inclusion; $B \leqslant A$ in $\Sigma$ if $B \supseteq A$ in $W$. The poset $\Sigma$ is a labelled chamber simplicial complex (see [2, Chap. III, § 1]).

A *chamber* of $\Sigma$ is a singleton $\{w\}$ with $w \in W$. A *vertex* of $\Sigma$ is a special coset $wW_{S-\{s\}}$ with $w \in W$ and $s \in S$. The face of $\Sigma$ of dimension $-1$ is the

coset $1 . W = W$ (this face has 0 vertices). The *fundamental chamber* of $\Sigma$ is $\{1\}$.

The Coxeter group $W$ naturally acts on $\Sigma$ by

$$w(vW_X) = (wv)W_X , \tag{2.1}$$

where $w \in W$, and $vW_X$ is a face of $\Sigma$ (i.e. a special coset).

The map which associates to a face $F = wW_X$ the subset $\lambda(F) = S - X$ of $S$ determines a labelling on $\Sigma$, called the *canonical labeling* of $\Sigma$, where $\lambda(F)$ is the *type* of a face $F$.

Two chambers $\{w\} \neq \{w'\}$ are *adjacent* if they have a common codimension 1 face, namely, if there exists an $s \in S$ such that $w' = ws$. A *gallery of length d* is a sequence $\{C_i\}_{i=0}^{d}$ of $d + 1$ chambers such that $C_i$ and $C_{i+1}$ are adjacent for $i = 0, 1, \ldots, d - 1$. In fact, to give a gallery $\{C_i\}_{i=0}^{d}$ is equivalent to give a source chamber $C_0$ and a sequence $s_1, \ldots, s_d$ of elements of $S$; the equivalence is given by $C_i = s_i \ldots s_2 s_1(C_0)$. A gallery $\{C_i\}_{i=1}^{d}$ joining two chambers $C_0$ and $C_d$ is called *minimal* if there is no gallery joining $C_0$ and $C_d$ with a smaller length.

The *distance* $d(C, D)$ between two chambers $C$ and $D$ is the length of a minimal gallery joining $C$ and $D$. We can easily see that, if $C = \{w\}$ and $D = \{v\}$, then

$$d(C, D) = l(w^{-1}v) . \tag{2.2}$$

The *distance* $d(C, F)$ between a chamber $C$ and a face $F$ of $\Sigma$ is

$$d(C, F) = \min\{d(C, D) \mid D \text{ a chamber having } F \text{ as face}\} . \tag{2.3}$$

As in (2.2), if $C = \{w\}$ and $F = vW_X$, then

$$d(C, F) = \min\{l(u) \mid u \in w^{-1}vW_X\} . \tag{2.4}$$

The *complex growth series* of a Coxeter system $(W, S)$ is the formal series in two variables

$$W_S(t_1, t_2) = \sum_F t_1^{d(C_0, F)} t_2^{\operatorname{codim}(F)} , \tag{2.5}$$

where the sum is over all the faces $F$ of $\Sigma$, and where $C_0 = \{1\}$ is the fundamental chamber.

Before stating and proving Propositions 1 and 2 and the Main Theorem, we are going to state two known results (Lemmas 1 and 2). A proof of Lemma 1 can be found either in [1, §4.1, exercise 3] or in [3, Lemma 1]. A proof of Lemma 2 can be found in [2, Chap. IV, §6].

Let $X \subseteq S$ be a subset and let $v \in W$. The element $v$ is called *X-minimal* if $v$ is of minimal length among the elements of $vW_X$.

LEMMA 1. *Let $X \subseteq S$ be a subset and let $v \in W$ be an X-minimal element of $W$. Then*

i) *$v$ is the unique X-minimal element of $vW_X$,*

ii) *for every $w = vu \in vW_X$, with $u = v^{-1}w \in W_X$, one has $l(w) = l(v) + l(u)$.*

For an integer $d \geqslant 0$, we denote by $\Sigma_d$ the subcomplex of $\Sigma = \Sigma(W, S)$ generated by the chambers $C$ of $\Sigma$ at distance $\leqslant d$ of $C_0 = \{1\}$.

$$\Sigma_d = \bigcup_F F ,$$

where the union is over all the faces $F$ of $\Sigma$ such that $d(C_0, F) \leqslant d$. We denote by $|\Sigma_d|$ the geometric realization of $\Sigma_d$.

LEMMA 2. i) *Let $(W, S)$ be a finite Coxeter system. Set $m = \max_{w \in W} l(w)$. Then $|\Sigma_d|$ is contractible if $d < m$, and $|\Sigma_d|$ is homotopic to the sphere $\mathbf{S}^{|S|-1}$ of dimension $|S| - 1$ if $d \geqslant m$.*

ii) *Let $(W, S)$ be an infinite Coxeter system. Then $|\Sigma_d|$ is contractible.*

PROPOSITION 1. *Let $(W, S)$ be a Coxeter system. Then*

$$W_S(t_1, 0) = W_S(t_1) \quad \text{and} \tag{2.6}$$

$$W_S(0, t_2) = (1 + t_2)^{|S|} . \tag{2.7}$$

*Proof.*

$$W_S(t_1, 0) = \sum_F t_1^{d(C_0, F)} ,$$

where the sum is over all the faces of $\Sigma$ of codimension 0, i.e. over all the chambers of $\Sigma$. Furthermore, if $F = C = \{w\}$, then, by (2.2), $d(C_0, F) = l(w)$. It follows that

$$W_S(t_1, 0) = \sum_{w \in W} t_1^{l(w)} = W_S(t_1) .$$

Now,

$$W_S(0, t_2) = \sum_F t_2^{\operatorname{codim}(F)} ,$$

where the sum is over all the faces $F$ of $\Sigma$ at distance 0 of $C_0$, i.e. over all the faces of $C_0$. Since $C_0$ is an $|S|-1$ dimensional simplex, it has $\binom{|S|}{i}$ faces of dimension $i$ (where $i = 0, 1, \ldots, |S|$). It follows that

$$W_S(0, t_2) = \sum_{i=0}^{|S|} \binom{|S|}{i} t_2^i = (1+t_2)^{|S|} . \quad \square$$

PROPOSITION 2. i) *Let* $(W, S)$ *be a finite Coxeter system. Then*

$$W_S(t_1, -1) = t_1^m , \tag{2.8}$$

*where* $m$ *is the maximal length in* $W$.

ii) *Let* $(W, S)$ *be an infinite Coxeter system. Then*

$$W_S(t_1, -1) = 0 . \tag{2.9}$$

*Proof.* Recall that $\Sigma_d$ is the subcomplex of $\Sigma$ generated by the chambers of $\Sigma$ at distance $\leqslant d$ of $C_0$, and that $|\Sigma_d|$ is the geometric realization of $\Sigma_d$. We denote by $E(|\Sigma_d|)$ the Euler characteristic of $|\Sigma_d|$. It is well known that $E(|\Sigma_d|)$ can be computed as follows:

$$\begin{aligned}(-1)^{|S|-1}E(|\Sigma_d|) &= (-1)^{|S|-1} \sum_{\substack{d(C_0,F)\leqslant d \\ F\neq W}} (-1)^{\dim(F)} \\ &= \sum_{\substack{d(C_0,F)\leqslant d \\ F\neq W}} (-1)^{\operatorname{codim}(F)} .\end{aligned}$$

On the other hand, we have

$$\begin{aligned}W_S(t_1, -1) &= \sum_F t_1^{d(C_0,F)} (-1)^{\operatorname{codim}(F)} \\ &= \sum_{d=0}^{\infty} \left( \sum_{d(C_0,F)=d} (-1)^{\operatorname{codim}(F)} \right) t_1^d .\end{aligned}$$

Thus

$$(-1)^{|S|-1}(E(|\Sigma_d|) - E(|\Sigma_{d-1}|)) \tag{2.10}$$

is the coefficient of $t_1^d$ in $W_S(t_1, -1)$ for $d \geqslant 1$, and

$$(-1)^{|S|-1}E(|\Sigma_0|) + (-1)^{|S|} \tag{2.11}$$

is the coefficient of $t_1^0$ in $W_S(t_1, -1)$. Lemma 2 implies that, if $(W, S)$ is a finite Coxeter system, then

$$E(|\Sigma_d|) = \begin{cases} 1 & \text{if} \quad d < m\,, \\ 1 + (-1)^{|S|-1} & \text{if} \quad d \geqslant m\,, \end{cases}$$

where $m$ is the maximal length in $W$; and if $(W, S)$ is an infinite Coxeter system, then

$$E(|\Sigma_d|) = 1\,,$$

for all $d \geqslant 0$. Replacing $E(|\Sigma_d|)$ by its value in (2.10) and (2.11), we obtain the equalities (2.8) and (2.9). □

MAIN THEOREM. *Let* $(W, S)$ *be a Coxeter system. Then*

$$W_S(t_1, t_2) = \sum_{X \subseteq S} t_2^{|X|} \frac{W_S(t_1)}{W_X(t_1)}\,. \tag{2.12}$$

*Proof.* Recall that the map which associates to a face $F = wW_X$ the subset $\lambda(F) = S - X$ of $S$ determines a labelling on $\Sigma$, where $\lambda(F)$ is the type of the face $F$. Clearly, if $\lambda(F) = Y$, then $\dim(F) = |Y| - 1$ and $\operatorname{codim}(F) = |S| - |Y| = |S - Y|$. Therefore

$$W_S(t_1, t_2) = \sum_{Y \subseteq S} t_2^{|S-Y|} \left( \sum_{F \in \mathscr{F}_Y} t_1^{d(C_0, F)} \right)\,, \tag{2.13}$$

where $\mathscr{F}_Y$ is the set of faces of $\Sigma$ of type $Y$. Let us prove

$$\sum_{F \in \mathscr{F}_Y} t_1^{d(C_0, F)} = \frac{W_S(t_1)}{W_{S-Y}(t_1)}\,, \tag{2.14}$$

for every $Y \subseteq S$. The equalities (2.13) and (2.14) clearly imply (2.12).

Let $X = S - Y$. Recall that an element $v \in W$ is $X$-minimal if it is of minimal length in $vW_X$. Every face $F \in \mathscr{F}_Y$ can be written $F = vW_X$ with $v$ $X$-minimal (take any element of minimal length in $F$). By (2.4), we have

$$d(C_0, F) = l(v)\,.$$

Lemma 1 shows that, for every $F \in \mathscr{F}_Y$, there is an unique $X$-minimal element $v$ in $F$. Therefore

$$\sum_{F \in \mathscr{F}_Y} t_1^{d(C_0, F)} = \sum_{v \in A_X} t_1^{l(v)}\,, \tag{2.15}$$

where $A_X$ is the set of all the $X$-minimal elements of $W$. Finally, Lemma 1 shows

$$\begin{aligned} W_S(t_1) &= \sum_{w \in W} t_1^{l(w)} \\ &= \sum_{v \in A_X} \sum_{w \in vW_X} t_1^{l(w)} \quad \text{(Lemma 1.i)} \\ &= \sum_{v \in A_X} \sum_{u \in W_X} t_1^{l(v)+l(u)} \quad \text{(Lemma 1.ii)} \\ &= \left( \sum_{v \in A_X} t_1^{l(v)} \right) W_X(t_1) \, . \end{aligned}$$

This and (2.15) imply (2.14) □

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