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ANOTHER ELEMENT

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COROLLARY 2 (C. Jordan [3]). A primitive subgroup of Sym(n) containing a transposition is all of Sym(n).

Proof. Let \mathcal{H} be a primitive subgroup of $\operatorname{Sym}(n)$ and τ a transposition in \mathcal{H} . Then \mathcal{H} permutes the components Γ_i of $\Gamma(\mathcal{H}, \tau)$ and so the vertex sets V_i of the Γ_i are permuted by \mathcal{H} . The primitivity of \mathcal{H} implies that the set $\{1, 2, \dots, n\}$ can be partitioned into disjoint subsets permuted by \mathcal{H} only if each subset has order one or there is just one subset of order n. Since the vertex set of Γ_i has more than one element, there is only one component and $\mathcal{H} = \operatorname{Sym}(n)$ by Corollary 1.

2. An application to Galois theory

We extend the theorem mentioned in the introduction replacing the condition that the degree of the polynomial be a prime greater than 3 by the condition that the degree of the polynomial be divisible only by primes greater than 3.

THEOREM 2. Left f(x) be a polynomial of degree n with rational coefficients and irreducible over the rational field. Assume that f(x) has exactly n-2 real roots. If n is divisible only by primes greater than 3 then the Galois group of the splitting field of f(x) is not solvable and f(x) is not solvable by radicals.

Proof. Let \mathcal{H} be the Galois group of f(x) over the rational field. We view \mathcal{H} as a permutation group on the n roots of f. Then complex conjugation, τ , is a transposition in \mathcal{H} of the two nonreal roots. Since f(x) is irreducible, \mathcal{H} is transitive on the set of n roots. By theorem 1, \mathcal{H} contains a subgroup isomorphic to the direct product of t copies of $\operatorname{Sym}(k)$ where tk = n. Since t is a divisor of t and t and t and t and t and t are in the hypothesis on the divisors of t implies t and t are in the subgroup. Thus t is not solvable by radicals.

3. Two generator subgroups of Sym(n)

Next we apply Theorem 1 to determine the subgroup of Sym(n) generated by a transposition and one other element. We first consider the case in which

the other element is an *n*-cycle. Let $\sigma = (1, 2, \dots, n)$ and $\tau = (a, b)$ with $1 \le a < b \le n$ and let $G = \langle \sigma, \tau \rangle$ be the group generated by the two elements. Then G is transitive on $\{1, 2, \dots, n\}$ because the cyclic subgroup $\langle \sigma \rangle$ is transitive. Theorem 1 will be applied to prove the following result.

Theorem 3. Let σ be an n-cycle and $\tau = (a, b)$ a transposition in $\operatorname{Sym}(n)$ and G the subgroup of $\operatorname{Sym}(n)$ generated by σ and τ . Let q be a positive integer such that $\sigma^q(a) = b$ and let $t = \gcd(n, q)$. Then t is the least positive integer such that τ and $\sigma^t \tau \sigma^{-t}$ correspond to edges in the same connected component of the graph $\Gamma(G, \tau)$ defined above. If we write n = tk for some integer k then G contains a normal subgroup S isomorphic to the direct product of t copies of $\operatorname{Sym}(k)$. The quotient G/S is cyclic of order t. In particular G is a solvable group if and only if $k \leq 4$.

Proof. Let S be the subgroup of G generated by all the transpositions conjugate in G to τ . By Theorem 1, S is the direct product of t copies of Sym(k) where t is the number of components of the graph $\Gamma(G, \tau)$. Let $\Gamma_1, \dots, \Gamma_t$ be the components of $\Gamma(G, \tau)$. Since σ is an n-cycle, the cyclic group $\langle \sigma \rangle$ permutes the components transitively. It follows that σ^t fixes each Γ_i and so $\sigma^t \in S$ and no smaller positive power of σ fixes any one of the Γ_i . Thus t is the least positive integer such that the edges corresponding to τ and $\sigma^t \tau \sigma^{-t}$ lie in the same component of $\Gamma(G, \tau)$. The fact that G/S is cyclic follows from the fact that G is generated by σ and τ and τ is in S. Thus G/S is generated by the coset σS .

The group G is solvable if and only if S and G/S are solvable; G/S is cyclic, hence solvable. S is solvable if and only if Sym(k) is solvable. It is well known that Sym(k) is solvable if and only if $k \le 4$.

We must now show that t is obtained as stated. We make a change of notation to facilitate the proof. Let R denote the ring Z/(n) of integers modulo n and view $\operatorname{Sym}(n)$ as a group of permutations of R. By renaming the elements, we may assume that σ is the n-cycle defined by $\sigma(x) = x + 1$ (with the addition in R used, of course). Let $\tau = (a, b)$ with $a, b \in R$ and take q = b - a. Since $\sigma^q(a) = a + q = b$, any other integer power of σ that carries a to b will have exponent congruent modulo n to b - a so there is no harm in assuming q = b - a.

Let $G = \langle \sigma, \tau \rangle$; we will show that the connected components of the graph $\Gamma(G, \tau)$ have the cosets x + qR as the vertex sets. The case in which qR has only two elements is somewhat exceptional and easy so we treat it first. When qR has two elements then n is even and $q \equiv n/2 \pmod{n}$ and

$$a + qR = a + (b - a)R = \{a, b\}.$$

Thus τ fixes every coset x + qR and σ carries x + qR to x + 1 + qR. Thus the edges of $\Gamma(G, \tau)$ are the pairs in the distinct cosets and each connected component consists of two vertices and one edge. There are n/2 components and so the number t of Theorem 3 is t = n/2 which equals gcd(n, q) as required.

Let r be the number of elements in qR and now assume r > 2. Thus r = n/gcd(n,q) and rq = 0 in R. The elements in a coset u + qR have the form u + jq, with $1 \le j \le r$. The cosets are permuted transitively by $\langle \sigma \rangle$. Each coset is left invariant by τ . This is clear for cosets not containing a or b. Since a + q = b, both a and b lie in a + qR so τ also leaves a + qR invariant. The edges of Γ are generated by applying the elements of G to the edge $\{a, b\}$. Thus the endpoints of an edge of Γ lie in the same coset of qR. Hence a connected component has all its vertices in one coset and thus a component has at most r vertices. Now we show that all vertices in a coset are connected. It is sufficient to show this for the coset a + qR since G is transitive on the components. The following computation is crucial for this verification:

(2)
$$(\tau \sigma^q)^j \{a, b\} = \{a, b + jq\} \quad \text{for} \quad 1 \le j \le r - 2.$$

We verify this by induction on j. For j = 1 we have

$$\tau \sigma^{q} \{a, b\} = \tau \{a + q, b + q\} = \tau \{b, b + q\}.$$

If we had b+q=a, then 0=b-a+q=2q and it follows that qR has only two elements. In the present case we have r>2 so $b+q\neq a$ and $\tau(b+q)=b+q$. Since $\tau(b)=a$ we see that (2) holds for j=1. Now assume (2) holds for j and that $j+1\leqslant r-2$. Then

$$(\tau \sigma^{q})^{j+1} \{a, b\} = \tau \sigma^{q} \{a, b + jq\}$$

$$= \tau \{a + q, b + (j+1)q\}$$

$$= \tau \{b, b + (j+1)q\}.$$

If b + (j + 1)q = a then (j + 2)q = 0. This implies $j + 2 \ge r$ contrary to the choices of j. Thus $\tau(b + (j + 1)q) = b + (j + 1)q$ and $\tau(b) = a$; thus (2) holds.

This computation shows that there are r-2 edges connecting a to vertices b+jq. The edge $\{a,b\}$ is not counted among these. Thus we account for r-1 edges containing a and r vertices in the connected component containing a. We have already seen that the components contain no more than r vertices. Hence there are exactly $r=n/\gcd(n,q)$ vertices in a component and the number of components is $n/r=\gcd(n,q)$ as we wanted to prove.

The group $\langle \sigma, \tau \rangle$ equals Sym(n) precisely when the graph Γ has just one component, that is t = 1 in Theorem 3. We have the following easily applied criterion.

COROLLARY 4. Let σ be an n-cycle and $\tau = (a, b)$ a transposition in $\operatorname{Sym}(n)$. Let q be an integer such that $\sigma^q(a) = b$. Then the group generated by σ and τ is all of $\operatorname{Sym}(n)$ if and only if $\gcd(n, q) = 1$.

We give two examples that determine the two generator groups using Theorem 3.

Example 1. Let $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$ and $\tau = (1, 5)$. The description of $\Gamma = \Gamma(\langle \sigma, \tau \rangle, \tau)$ may be obtained using Theorem 3. Since $\sigma^4(1) = 5$ we find there are $t = \gcd(8, 4) = 4$ components with 2 vertices in each.

In order to determine the group $G = \langle \sigma, \tau \rangle$ explicitly, we find the component of Γ . We find the edges of Γ by repeatedly applying σ to the edge $\{1, 5\}$ to obtain the edges

$$\{2,6\},\{3,7\},\{4,8\},\{1,5\}.$$

Application of τ does not yield any new edges and so these are all the edges in Γ . The groups of permutations of the components are:

$$S_1 = \langle (2,6) \rangle, \quad S_2 = \langle (3,7) \rangle, \quad S_3 = \langle (4,8) \rangle, \quad S_4 = \langle (1,5) \rangle.$$

The conjugation action of σ is to cyclically permute the factors S_1 , S_2 , S_3 , S_4 and $\sigma^4 = (1, 5)(2, 6)(3, 7)(4, 8)$ is in $S_1 \times \cdots \times S_4$. Thus the order of G is

$$|S_1|^4 |\langle \sigma \rangle / \langle \sigma^4 \rangle| = 2^4 \cdot 4 = 64$$
.

Example 2. Let $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$ and $\tau = (1, 6)$. Since $\sigma^{5}(1) = 6$ and gcd(8, 5) = 1, Corollary 4 implies $\langle \sigma, \tau \rangle = \text{Sym}(8)$.

Now we consider the description of $\langle \sigma, \tau \rangle$ with τ a transposition and σ any element of Sym(n), not necessarily an n-cycle. The discussion will be broken into cases depending on how σ and τ are realted.

To make the notation simpler, let us assume $\tau = (1, 2)$. We may express σ as a product of disjoint cycles

$$\sigma = \xi_1 \xi_2 \cdots \xi_r, \quad \xi_i \text{ a cycle }.$$

Let V_i be the set of symbols moved by ξ_i so that ξ_i permutes the elements of V_i transitively and fixes the elements of V_j for $j \neq i$.

The first case in which σ is a cycle and τ is a transposition moving two symbols that are also moved by σ is covered in Theorem 3.

Second case. $1, 2 \in V_1$. This is the case in which the two elements moved by τ are moved by a single cycle appearing in the decomposition of σ .

Since $\sigma(V_1) = V_1$ and $\tau(V_1) = V_1$, we obtain a homomorphism ρ of $G = \langle \sigma, \tau \rangle$ into $\operatorname{Sym}(V_1)$ defined by letting $\rho(\eta)$ be the restriction to V_1 of $\eta \in G$. Thus $\rho(\sigma) = \xi_1$ and $\rho(\tau) = \tau$. The group $\rho(G) = \langle \xi_1, \tau \rangle$ is determined by Theorem 3 since ξ_1 is a cycle on V_1 and τ is a transposition. The kernel of ρ is the set of elements in G that leave fixed each element of V_1 .

We will describe the kernel of ρ precisely but first we examine a potentially larger group containing G.

Let $\gamma = \xi_1^{-1} \sigma$ so that

$$\sigma = \xi_1 \xi_2 \cdots \xi_r = \xi_1 \gamma = \gamma \xi_1.$$

Of course ξ_1 need not be in G so γ need not be in G. Let \mathscr{G} be the group generated by σ , τ , and γ . Then we also have $\mathscr{G} = \langle \xi_1, \tau, \gamma \rangle$. The subgroup $\langle \xi_1, \tau \rangle$ of \mathscr{G} operates on V_1 while fixing each point in its complement and $\langle \gamma \rangle$ operates on the complement of V_1 while fixing each point of V_1 . It follows that the group \mathscr{G} is the direct product

$$\mathscr{G} = \langle \xi_1, \tau \rangle \times \langle \gamma \rangle$$
. (*)

The subgroup of \mathscr{G} fixing V_1 is $\langle \gamma \rangle$ and so the kernel of $\rho: G \to \langle \xi_1, \tau \rangle$ is the cyclic group $G \cap \langle \gamma \rangle$.

The subgroup S of $\langle \xi_1, \tau \rangle$ generated by all the conjugates of τ is actually a subgroup of G. To see this we note that any element η of G can be expressed as

$$\eta = \rho(\eta)\gamma^i$$
 for some integer *i*.

Thus

$$\eta \tau \eta^{-1} = \rho(\eta) \gamma^{i} \tau \gamma^{-i} \rho(\eta)^{-1} = \rho(\eta) \tau \rho(\eta)^{-1}.$$

Since ρ maps G onto $\langle \xi_1, \tau \rangle$ it follows that every conjugate of τ in $\langle \xi_1, \tau \rangle$ is also conjugate of τ in G and conversely. The subgroup generated by all these conjugates, denoted as S in Theorem 3, is contained in G and in the first factor of \mathcal{G} in (*).

We will factor out the normal subgroup S from both G and \mathscr{G} . Since $\tau \in S$ it follows that

$$\frac{\mathscr{G}}{S} \cong \langle \bar{\xi}_1 \rangle \times \langle \bar{\gamma} \rangle ,$$

$$\frac{G}{S} \cong \langle \bar{\sigma} \rangle = \langle \bar{\xi}_1 \bar{\gamma} \rangle ,$$

where $\bar{\eta}$ is the coset ηS . This factor will be used in two ways: We will determine the index of S in G and thereby determine the order of G and we will also determine the smallest power of γ that lies in G thereby finding the kernel of ρ .

We are dealing with a two-generator abelian group \mathcal{G}/S and the subgroup G/S generated by the product of the two generators. The first generator $\bar{\xi}_1$ has order t, the number of connected components of the graph $\Gamma(\xi_1, \tau)$. Let g denote the order of γ . Note that g is also the order of $\bar{\gamma}$ because $S \cap \langle \gamma \rangle = e$. Then the order of $\bar{\sigma} = \bar{\xi}_1 \bar{\gamma}$ is the least common multiple of t and g, denoted as [t, g]. Thus the order of G is the order of G times G0 and G1 is the order of G2. The order of G3 times G3 is the order of G4 times G5. The order of G6 onto this group. Hence the kernel of G6 has order

$$|\ker \rho| = \frac{|S|[t,g]}{|S|t} = \frac{[t,g]}{t} = \frac{g}{(t,g)},$$

where (t, g) is the greatest common divisor of t and g. Since the order of γ^t is g/(t, g) it follows that γ^t generates the kernel of ρ ; we have $G \cap \langle \gamma \rangle = \langle \gamma^t \rangle$.

We summarize this case in a theorem.

Theorem 5. Suppose $\sigma = \xi_1 \xi_2 \cdots \xi_r$ is the cycle decomposition of σ and $\tau = (a, b)$ is a transposition with both a and b moved by the cycle ξ_1 appearing in σ . Let $G = \langle \sigma, \tau \rangle$. Let $\gamma = \xi_1^{-1} \sigma$ and let n be the order of ξ_1, g the order of γ and t the number of connected components of the graph $\Gamma(\langle \xi_1, \tau \rangle, \tau)$ and k = n/t. Then the subgroup S of G generated by all the G-conjugates of τ is isomorphic to the direct product of t copies of Sym(k). The quotient group G/S is cyclic with order [t, g], the least common multiple of t and g. The order of G is $(k!)^t[t, g]$. The homomorphism $\rho: G \to \langle \xi_1, \tau \rangle$ defined by restricting the action of G to the set of symbols moved by ξ_1 has kernel $\langle \gamma^t \rangle$.

Example 3. This example illustrates the ideas used in the proof of Theorem 5. Let $\sigma = (1, 2, 3, 4, 5, 6) (7, 8, 9)$ and $\tau = (1, 3)$. Then $\xi_1 = (1, 2, 3, 4, 5, 6)$ and $\gamma = (7, 8, 9)$ in the notation of Theorem 5. We first describe the group $\langle \xi_1, \tau \rangle$ using Theorem 3 and the graph $\Gamma = \Gamma(\langle \xi_1, \tau \rangle, \tau)$. The lowest power of ξ_1 that has the same effect as τ on 1 is ξ_1^2 . Thus the number of components of Γ is t = gcd(6, 2) = 2. Thus the components of Γ have vertex sets $\{1, 3, 5\}$ and $\{2, 4, 6\}$ as we find by applying

powers of ξ_1 to $\{1, 3\}$. Thus the subgroup generated by the G-conjugates of r is $S = S_1 \times S_2$ with each $S_i \cong \text{Sym}(3)$.

The group $G = \langle \sigma, \tau \rangle$ admits a homomorphism ρ onto $\langle \xi_1, \tau \rangle$ defined by restriction of elements of G to the action induced on $\{1, 2, 3, 4, 5, 6\}$, the set moved by ξ_1 . The kernel of ρ is the subgroup of G fixing the symbols 1, 2, 3, 4, 5, 6. The kernel was shown to be $G \cap \langle \gamma \rangle = \langle \gamma^t \rangle$. Since t = 2 and $\gamma = (7, 8, 9)$ has order 3, it follows that the kernel of ρ is the group $\langle \gamma \rangle$ of order 3. The group G must also contains $\xi_1 = \gamma^{-1}\sigma$ and so we have the decomposition

$$G = \langle \sigma, \tau \rangle = \langle (1, 2, 3, 4, 5, 6)(7, 8, 9), (1, 3) \rangle$$

= $\langle \xi_1, \tau \rangle \times \langle \gamma \rangle = \langle (1, 2, 3, 4, 5, 6), (1, 3) \rangle \times \langle (7, 8, 9) \rangle.$

The order of G is $(3!) \cdot 2 \cdot 3 = 6^3$.

If this example is changed by letting $\sigma = (1, 2, 3, 4, 5, 6)(7, 8)$, so that $\gamma = (7, 8)$, but keeping the same τ then t is unchanged and so the kernel of ρ is $\langle \gamma^2 \rangle = e$. Thus $\rho: G \to \langle \xi_1, \tau \rangle$ is an isomorphism. The order of G is $(3!)^2 \cdot 2$.

The two cases covered by Theorems 3 and 5 take care of the difficult cases. All the remaining cases can be handled quickly.

Third Case. $\tau = (1, 2)$ and $\sigma(1) = 1$ and $\sigma(2) = 2$; i.e. σ fixes the two symbols moved by τ . Then

$$G = \langle \sigma, \tau \rangle = \langle \sigma \rangle \times \langle \tau \rangle$$

is the direct product of two cyclic groups.

Fourth Case. $\tau = (1, 2)$ and $\sigma = (1, a_2, \dots, a_r)$ $(2, b_2, \dots, b_s)\gamma$ where $r \ge 1$, $s \ge 1$; i.e. σ moves at least one of the symbols moved by τ and if it moves both, they do not appear in the same cycle of σ . If r = 1 then $\sigma(1) = 1$; similarly for s = 1. If r = s = 1 then we are in the third case so we may assume either r or s is greater than 1. It is assumed that this is the cycle decomposition of σ and that γ is the product of the disjoint cycles not moving 1 or 2. Then we let σ_1 be the element

$$\sigma_1 = \sigma \tau = (1, a_2, \dots, a_r)(2, b_2, \dots, b_s)\gamma(1, 2)$$

= $(1, b_2, \dots, b_s, 2, a_2, \dots, a_r)\gamma$.

Since the group generated by σ and τ is the same as the group generated by σ_1 and τ , we may replace σ by σ_1 . We are back in the first case now because both 1 and 2 are moved by the same cycle appearing in the generator σ_1 .

We may collect the results as follows.

SUMMARY. Let $G = \langle \sigma, \tau \rangle$ with $\sigma, \tau \in \text{Sym}(n)$ and τ a transposition.

- 1. If σ is an *n*-cycle, the G is described in Theorem 3.
- 2. If σ is a product of disjoint cycles, one of which moves both the symbols moved by τ , then G is described in Theorem 5.
- 3. If σ fixes both symbols moved by τ then $G = \langle \sigma \rangle \times \langle \tau \rangle$ is an abelian group.
- 4. If σ moves one, but not both of, the symbols moved by τ or if σ moves both symbols moved by τ but not in the same cycle then σ may be replaced by $\sigma_1 = \tau \sigma$ and then $G = \langle \sigma_1, \tau \rangle$ and G is described as in case 1 or 2.

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