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Proof. Suppose $\mathrm{cat}_0(\Lambda Z) = m$. Then ΛZ is a retract of $\Lambda Z/\Lambda^{>m}Z$ and we see that $\Lambda Z \otimes \Lambda y$ is a retract of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$. Now, the maximal product length of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$ is $m+1$ and this is sufficient to ensure $\mathrm{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant m+1$. □

Now, by induction, we see that $\mathrm{cat}_0(\Lambda) \leqslant n$ (since for x_1 of odd degree $\mathrm{cat}_0(\Lambda x_1) = 1$). Putting this together with the Lemma gives

THEOREM 2. *If* $\Lambda = (\Lambda(x_1, \cdots x_n), d)$ *with* $\deg(x_i) = \text{odd}$ *for each* i, *then* $\mathrm{cat}_0(\Lambda) = n$.

This result may be applied, for example, to a manifold obtained as an iterated principal bundle. That is, for compact Lie groups G_i, $i = 1$ to N.

$M_1 = G_1$; M_i is obtained from M_{i-1} as a principal G_i-bundle over M_{i-1}.

$M = M_N$

Each G_i is, rationally, a product of $\mathrm{rank}(G_i)$ odd spheres, so the minimal model of M has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_s), d)$$

with $\deg(x_i) = \text{odd}$ and $s = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. $\mathrm{cat}_0(M) = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. *If* M *is an iterated principal bundle with fibres* G_i, *then the number of critical points of any smooth function on* M *is bounded below by* $\sum_i \mathrm{rank}(G_i) + 1$.

Note that we have not determined $\mathrm{cat}(M)$, so the true effectiveness of Lusternik-Schnirelmann theory may not have been exploited.

§6. GANEA'S CONJECTURE

The Ganea Conjecture states that, for a finite CW complex X, $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$ for any sphere S^k. Although unproven in general, various cases of the conjecture have been shown to be true. We add nilmanifolds to that list:

THEOREM. *Ganea's Conjecture is true for nilmanifolds.*

Proof. Let M be a nilmanifold. Then

$$\begin{aligned}\dim M + 1 &= e_0(M) + 1 \\ &= e_0(M \times S^k) \text{ since } e_0 \text{ respects products} \\ &\leqslant \operatorname{cat}(M \times S^k) \\ &\leqslant \operatorname{cat}(M) + 1 \text{ Fox's inequality} \\ &= \dim M + 1 .\end{aligned}$$

Hence all inequalities are equalities and $\operatorname{cat}(M \times S^k) = \operatorname{cat}(M) + 1$. □

ADDED IN PROOF. By using the equality $e_0(M) = \dim(M)$ and extending the e_0-invariant to maps, C. McCord and the author have given a proof of the Arnold Conjecture for nilmanifolds (cf. C. McCord and J. Opera, *Rational Ljusternik-Schnirelmann Category and the Arnold Conjecture for Nilmanifolds,* preprint 1992). That is, any smooth 1-periodic Hamiltonian system on a symplectic nilmanifold M has at least $\dim(M) + 1$ contractible 1-periodic orbits.

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