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Proof. Suppose $\text{cat}_0(\Lambda Z) = m$. Then ΛZ is a retract of $\Lambda Z/\Lambda^{>m}Z$ and we see that $\Lambda Z \otimes \Lambda y$ is a retract of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$. Now, the maximal product length of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$ is $m + 1$ and this is sufficient to ensure $\text{cat}_0(\Lambda Z \otimes \Lambda y) \leq m + 1$. \square

Now, by induction, we see that $\text{cat}_0(\Lambda) \leq n$ (since for x_1 of odd degree $\text{cat}_0(\Lambda x_1) = 1$). Putting this together with the Lemma gives

THEOREM 2. *If $\Lambda = (\Lambda(x_1, \dots, x_n), d)$ with $\deg(x_i) = \text{odd}$ for each i , then $\text{cat}_0(\Lambda) = n$.*

This result may be applied, for example, to a manifold obtained as an iterated principal bundle. That is, for compact Lie groups $G_i, i = 1$ to N .

$M_1 = G_1$; M_i is obtained from M_{i-1} as a principal G_i -bundle over M_{i-1} .

$M = M_N$

Each G_i is, rationally, a product of $\text{rank}(G_i)$ odd spheres, so the minimal model of M has the form,

$$\Lambda(M) = (\Lambda(x_1, \dots, x_s), d)$$

with $\deg(x_i) = \text{odd}$ and $s = \sum_{i=1}^N \text{rank}(G_i)$.

COROLLARY. $\text{cat}_0(M) = \sum_{i=1}^N \text{rank}(G_i)$.

COROLLARY. *If M is an iterated principal bundle with fibres G_i , then the number of critical points of any smooth function on M is bounded below by $\sum_i \text{rank}(G_i) + 1$.*

Note that we have not determined $\text{cat}(M)$, so the true effectiveness of Lusternik-Schnirelmann theory may not have been exploited.

§6. GANEA'S CONJECTURE

The Ganea Conjecture states that, for a finite CW complex X , $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for any sphere S^k . Although unproven in general, various cases of the conjecture have been shown to be true. We add nilmanifolds to that list:

THEOREM. *Ganea's Conjecture is true for nilmanifolds.*

Proof. Let M be a nilmanifold. Then

$$\begin{aligned} \dim M + 1 &= e_0(M) + 1 \\ &= e_0(M \times S^k) \text{ since } e_0 \text{ respects products} \\ &\leq \text{cat}(M \times S^k) \\ &\leq \text{cat}(M) + 1 \text{ Fox's inequality} \\ &= \dim M + 1 . \end{aligned}$$

Hence all inequalities are equalities and $\text{cat}(M \times S^k) = \text{cat}(M) + 1$. \square

ADDED IN PROOF. By using the equality $e_0(M) = \dim(M)$ and extending the e_0 -invariant to maps, C. McCord and the author have given a proof of the Arnold Conjecture for nilmanifolds (cf. C. McCord and J. Opera, *Rational Ljusternik-Schnirelmann Category and the Arnold Conjecture for Nilmanifolds*, preprint 1992). That is, any smooth 1-periodic Hamiltonian system on a symplectic nilmanifold M has at least $\dim(M) + 1$ contractible 1-periodic orbits.

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