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*Proof.* Suppose  $\text{cat}_0(\Lambda Z) = m$ . Then  $\Lambda Z$  is a retract of  $\Lambda Z/\Lambda^{>m}Z$  and we see that  $\Lambda Z \otimes \Lambda y$  is a retract of  $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$ . Now, the maximal product length of  $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$  is  $m + 1$  and this is sufficient to ensure  $\text{cat}_0(\Lambda Z \otimes \Lambda y) \leq m + 1$ .  $\square$

Now, by induction, we see that  $\text{cat}_0(\Lambda) \leq n$  (since for  $x_1$  of odd degree  $\text{cat}_0(\Lambda x_1) = 1$ ). Putting this together with the Lemma gives

**THEOREM 2.** *If  $\Lambda = (\Lambda(x_1, \dots, x_n), d)$  with  $\deg(x_i) = \text{odd}$  for each  $i$ , then  $\text{cat}_0(\Lambda) = n$ .*

This result may be applied, for example, to a manifold obtained as an iterated principal bundle. That is, for compact Lie groups  $G_i$ ,  $i = 1$  to  $N$ .

$M_1 = G_1$ ;  $M_i$  is obtained from  $M_{i-1}$  as a principal  $G_i$ -bundle over  $M_{i-1}$ .

$M = M_N$

Each  $G_i$  is, rationally, a product of  $\text{rank}(G_i)$  odd spheres, so the minimal model of  $M$  has the form,

$$\Lambda(M) = (\Lambda(x_1, \dots, x_s), d)$$

with  $\deg(x_i) = \text{odd}$  and  $s = \sum_{i=1}^N \text{rank}(G_i)$ .

**COROLLARY.**  $\text{cat}_0(M) = \sum_{i=1}^N \text{rank}(G_i)$ .

**COROLLARY.** *If  $M$  is an iterated principal bundle with fibres  $G_i$ , then the number of critical points of any smooth function on  $M$  is bounded below by  $\sum_i \text{rank}(G_i) + 1$ .*

Note that we have not determined  $\text{cat}(M)$ , so the true effectiveness of Lusternik-Schnirelmann theory may not have been exploited.

## §6. GANEA'S CONJECTURE

The Ganea Conjecture states that, for a finite CW complex  $X$ ,  $\text{cat}(X \times S^k) = \text{cat}(X) + 1$  for any sphere  $S^k$ . Although unproven in general, various cases of the conjecture have been shown to be true. We add nilmanifolds to that list:

**THEOREM.** *Ganea's Conjecture is true for nilmanifolds.*

*Proof.* Let  $M$  be a nilmanifold. Then

$$\begin{aligned}
 \dim M + 1 &= e_0(M) + 1 \\
 &= e_0(M \times S^k) \text{ since } e_0 \text{ respects products} \\
 &\leqslant \text{cat}(M \times S^k) \\
 &\leqslant \text{cat}(M) + 1 \text{ Fox's inequality} \\
 &= \dim M + 1.
 \end{aligned}$$

Hence all inequalities are equalities and  $\text{cat}(M \times S^k) = \text{cat}(M) + 1$ .  $\square$

ADDED IN PROOF. By using the equality  $e_0(M) = \dim(M)$  and extending the  $e_0$ -invariant to maps, C. McCord and the author have given a proof of the Arnold Conjecture for nilmanifolds (cf. C. McCord and J. Opera, *Rational Ljusternik-Schnirelmann Category and the Arnold Conjecture for Nilmanifolds*, preprint 1992). That is, any smooth 1-periodic Hamiltonian system on a symplectic nilmanifold  $M$  has at least  $\dim(M) + 1$  contractible 1-periodic orbits.

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