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$$\begin{array}{ccccc} S^1 & \to & M & = & K(\pi, 1) \\ & & \downarrow & & \\ S^1 & \to & M_{n-1} & \overset{\tau_n}{\to} & \mathbf{C}P(\infty) \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ S^1 & \to & M_1 & \overset{\tau_2}{\to} & \mathbf{C}P(\infty) \\ & & \downarrow & & \\ & & * & \overset{\tau_1}{\to} & \mathbf{C}P(\infty) . \end{array}$$

We can assume (by finite dimensionality) that each τ_i has image in a finite $\mathbf{C}P(n)$, so thus may be approximated by a smooth map. Hence, each M_j is a compact manifold with

$$\dim(M_j) = \dim(M_{j-1}) + 1 .$$

Thus, $\dim(M) = \operatorname{rank}(\pi) = n$.

§4. Category of nilmanifolds

The decomposition of $M = K(\pi, 1)$ into a tower of principal S^1-bundles is, in fact, the Postnikov decomposition of M with k-invariants the τ_i. By the fundamental theorem of rational homotopy theory, the minimal model has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_n), d) , \quad \deg(x_i) = 1$$

with $dx_i = \tau_i$, where τ_i is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbf{Z})$. Note that $\Lambda(M)$ is an exterior algebra because all generators are in degree 1. Therefore, since $\dim M = n$, the only possibility for a cocycle representing the fundamental class is $x_1 \cdots x_n$. Hence, $e_0(M) = n$ and this immediately implies,

Proof of Theorem 1. $n = e_0(M) \leqslant \operatorname{cat}_0(M) \leqslant \operatorname{cat}(M) \leqslant \dim M = n$. □

Example. Consider the 3-dimensional Heisenberg group $U_3(\mathbf{R})$ and mod out by $U_3(\mathbf{Z})$. The resulting M is a 3-manifold obtained as a principal bundle,

$$S^1 \to M \to T^2$$

with classifying element (over the rationals) $xy \in H^2(T^2; \mathbf{Q})$, where x and y are one-dimensional generators. The minimal model of M is then given by

$$\Lambda(M) = \Lambda(x, y, z) \quad \deg(x) = \deg(y) = \deg(z) = 1$$

with $dx = 0 = dy$ and $dz = xy$. Additive generators for cohomology are then,

$H^1 : x, y$

$H^2 : xz, yz$ (Massey products!)

$H^3 : zyx$.

Note that $\mathrm{cup}(M) = 2$, but $\mathrm{cat}(M) = 3$.

In some sense then, the proof of Theorem 1 is simply an observation that the techniques of rational homotopy theory work particularly well for nilmanifolds.

PROBLEM. If π is not nilpotent, then a $K(\pi, 1)$ is not a nilpotent space, so the minimal model does not describe a "rational type". Is it possible, however, that enough information about a $K(\pi, 1)$ is present in the model to determine its category (in the compact case say)?

§5. HIGHER DEGREE ANALOGUES

An analogue of the minimal model of a nilmanifold is one of the form,

$$(\Lambda(x_1, \cdots x_n), d) , \quad \mathrm{degree}(x_i) = \mathrm{odd} .$$

Such an algebra is known to satisfy rational Poincaré duality (see [5]) and to have formal top dimension $\sum_i \deg(x_i)$. But, plainly, the same argument as before applies to show that the "only" element in this exterior algebra which can reach the stated dimension is $x_1 \cdots x_n$. Hence (since this is the longest product in Λ), the fundamental class is maximally represented by a product of length n and

LEMMA. $e_0(\Lambda) = n$.

Now, we may consider Λ as built up by adjoining odd generators one at a time (with decomposable differential). Let ΛZ be a minimal cdga and y of odd degree. Then

PROPOSITION. (See Theorem 4.7 and Lemma 6.6 of [3].)

$$\mathrm{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant \mathrm{cat}_0(\Lambda Z) + 1 .$$