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The remaining candidates are listed below, together with an indication in parenthesis showing that each one (except 505) is excluded by Theorem 2 in Section 2: if  $N$  has a prime factor  $p$  such that  $p^f \equiv -1 \pmod{N'}$ , where  $N'$  is the largest divisor of  $N$  relatively prime to  $p$ , then there is no (periodic) Barker sequence of length  $4N^2$ .

REMAINING CANDIDATES (excluded by Theorem 2, except  $N = 505$ .)

$N$		$N$	
$65 = 5 \cdot 13$	$(5^2 \equiv -1 \pmod{13})$	$425 = 5^2 \cdot 17$	$(5^8 \equiv -1 \pmod{17})$
$85 = 5 \cdot 17$	$(17^2 \equiv -1 \pmod{5})$	$445 = 5 \cdot 89$	$(89 \equiv -1 \pmod{5})$
$145 = 5 \cdot 29$	$(29 \equiv -1 \pmod{5})$	$481 = 13 \cdot 37$	$(37^6 \equiv -1 \pmod{13})$
$185 = 5 \cdot 37$	$(37^2 \equiv -1 \pmod{5})$	$485 = 5 \cdot 97$	$(97^2 \equiv -1 \pmod{5})$
$205 = 5 \cdot 41$	$(5^{10} \equiv -1 \pmod{41})$	$493 = 17 \cdot 29$	$(17^2 \equiv -1 \pmod{29})$
$221 = 13 \cdot 17$	$(13^2 \equiv -1 \pmod{17})$	$505 = 5 \cdot 101$	
$265 = 5 \cdot 53$	$(53^2 \equiv -1 \pmod{5})$	$533 = 13 \cdot 43$	$(43^3 \equiv -1 \pmod{13})$
$305 = 5 \cdot 61$	$(5^{15} \equiv -1 \pmod{61})$	$545 = 5 \cdot 109$	$(109 \equiv -1 \pmod{5})$
$325 = 5^2 \cdot 13$	$(5^2 \equiv -1 \pmod{13})$	$565 = 5 \cdot 113$	$(113^2 \equiv -1 \pmod{5})$
$365 = 5 \cdot 73$	$(73^2 \equiv -1 \pmod{5})$	$629 = 17 \cdot 37$	$(37^8 \equiv -1 \pmod{17})$
$377 = 13 \cdot 29$	$(13^7 \equiv -1 \pmod{29})$	$685 = 5 \cdot 137$	$(137^2 \equiv -1 \pmod{5})$

The case  $N = 505 = 5 \cdot 101$  cannot be excluded by Theorem 2, because  $101 \equiv 1 \pmod{5}$  and  $5^{25} \equiv 1 \pmod{101}$ . However, 505 can still be excluded by Turyn's Inequality, as observed in [JL]: choosing  $p = 101$  and  $w = 2 \cdot 101^2$ , so that  $p$  is trivially semi-primitive modulo  $w$ , we would have

$$p \leq \frac{v}{w} = 2 \cdot 5^2 = 50,$$

a contradiction to the assumed existence of a Barker sequence of length  $4 \cdot 505^2$ .

The first open case is thus  $N = 689 = 13 \cdot 53$ . We have  $53 \equiv 1 \pmod{13}$  and  $13^{13} \equiv 1 \pmod{53}$ , so that neither 53 is semi-primitive mod 13, nor 13 is semi-primitive mod 53. The next open case is  $N = 793 = 13 \cdot 61$ .

#### 4. THE USE OF THE MULTIPLIER THEOREM

In this section we give the details of some (typical) non-existence proofs needed to establish the tables, using the multiplier theorem.

Recall that if  $D$  is a cyclic difference set with parameters  $(v, k, \lambda)$ , and if  $n = k - \lambda$  is greater than  $\lambda$ , then the group of multipliers of  $D$  contains the intersection  $M$  in  $(\mathbf{Z}/v\mathbf{Z})^*$  of the subgroups generated by  $l_1, \dots, l_r$ , where  $l_1, \dots, l_r$  are the prime factors of  $n$ .

(1) *Parameters* ( $v = 181, k = 81, \lambda = 36$ ), *Table I with*  $t = 9$ .

Here,  $n = 3^2 \cdot 5$ , and since  $5 \equiv 3^6 \pmod{181}$ , the multiplier theorem says that if an abelian difference set exists with these parameters, then 5 is a multiplier. The orbits of the multiplication by 5 in  $\mathbf{Z}/181\mathbf{Z}$  are  $\{0\}$  and 12 orbits of cardinality 15, e.g.

$$\{1, 5, 25, 125, 82, 48, 59, 114, 27, 135, 132, 117, 42, 29, 145\}.$$

(Note that 181 is a prime number.) No subset of  $G = \mathbf{Z}/181\mathbf{Z}$  of cardinality  $k = 81$  may thus be a union of orbits.

(2) *Parameters* ( $v = 4901, k = 2401, \lambda = 1176$ ), *Table I with*  $t = 49$ .

Here,  $n = 5^2 \cdot 7^2$ . We have  $25 = 5^2 \equiv 7^6 \pmod{4901}$ . Therefore, if an abelian difference set exists,  $m = 25$  must be a multiplier. Writing the group  $G = \mathbf{Z}/4901\mathbf{Z}$  as  $G = \mathbf{Z}/13^2\mathbf{Z} \times \mathbf{Z}/29\mathbf{Z}$ , with group operation  $(a, b) \cdot (a', b') = (a + a', b + b')$ , the orbits under multiplication by  $m = 25$  are

$$E = \{(0, 0)\}$$

$$U_i = \{(13i, 0), (-13i, 0)\} \quad i = 1, 2, 3, 4, 5, 6$$

$$V_j = \{(j, 0), (25j, 0), (118j, 0), (77j, 0), (66j, 0), (129j, 0), (14j, 0), (12j, 0), (131j, 0), (64j, 0), (79j, 0), (116j, 0), (27j, 0), (-j, 0), \dots\}$$

$j = 1, \dots, 6$ , each  $V_j$  of cardinality 26.

$$X = \{(0, 1), (0, 25), (0, 16), (0, 23), (0, 24), (0, 20), (0, 7)\}$$

$$Y = \{(0, 2), (0, 21), (0, 3), (0, 17), (0, 19), (0, 11), (0, 14)\}$$

$$\bar{X} = \{(0, -x) \mid (0, x) \in X\}$$

$$\bar{Y} = \{(0, -y) \mid (0, y) \in Y\}$$

each of cardinality 7.

There are moreover, the 24 orbits  $U_i \cdot X, U_i \cdot \bar{X}, U_i \cdot Y, U_i \cdot \bar{Y}$  of cardinality 14, where

$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\}.$$

Finally, there are 24 orbits  $V_i \cdot X, V_i \cdot \bar{X}, V_i \cdot Y, V_i \cdot \bar{Y}$  of cardinality 182. Contrary to the preceding example, there are many ways of writing the cardinality 2401 of a putative difference set  $D$  as a sum of numbers taken from the set of orbit cardinalities.

To ease calculations, we view a subset  $S \subset G$  as the element  $\sum_{s \in S} s$  in the integral group ring. Note that, with this convention, the product  $S \cdot T$  in  $\mathbf{Z}G$  coincides with the element of  $\mathbf{Z}G$  associated with the product set

$S \cdot T = \{s \cdot t \mid s \in S, t \in T\}$ . A difference set  $D$ , if it exists with the above parameters, can be written as

$$D = C + AX + BY + P\bar{X} + Q\bar{Y}$$

where  $C$ , as well as  $A, B, P, Q$ , is of the form

$$C = \alpha E + \sum_{i=1}^6 \beta_i U_i + \sum_{j=1}^6 \gamma_j V_j$$

with coefficients  $\alpha, \beta_1, \dots, \beta_6, \gamma_1, \dots, \gamma_6$  all equal to 0 or 1.

As in Section 1,  $D$  is a difference set if and only if

$$D\bar{D} = 1225 + 1176 \cdot \left(1 + \sum_{i=1}^6 U_i + \sum_{j=1}^6 V_j\right) \cdot (1 + X + \bar{X} + Y + \bar{Y}).$$

Now, writing  $G = G_1 \times G_2$  as above,  $G_1 = \mathbf{Z}/13^2\mathbf{Z}$ ,  $G_2 = \mathbf{Z}/29\mathbf{Z}$ , let  $\pi: \mathbf{Z}G \rightarrow \mathbf{Z}G_1$  be the projection on the group ring of  $G_1$ . We have  $\pi X = \pi\bar{X} = \pi Y = \pi\bar{Y} = 7$ , and reducing modulo 7,

$$\pi(D\bar{D}) = C\bar{C} = 0 \text{ in } \mathbf{F}_7 G_1.$$

The involution of  $\mathbf{Z}G$ , sending  $(a, b)$  to  $(\overline{a, b}) = (-a, -b)$ , is the identity on  $U_i, V_j$ :

$$\bar{U}_i = U_i, \quad \bar{V}_j = V_j.$$

Therefore  $\bar{C} = C$  and  $C^2 = 0$  in  $\mathbf{F}_7 G_1$ . However,  $\mathbf{F}_7 G_1$ , where  $G_1$  is of order  $13^2$ , prime to 7, is a semi-simple algebra and does not contain any nilpotent element. It follows that  $C = 0$  in  $\mathbf{F}_7 G_1$ . Since the coefficients of  $C = \alpha E + \sum_{i=1}^6 \beta_i U_i + \sum_{j=1}^6 \gamma_j V_j$  are all 0 or 1, this implies  $C = 0$  in  $\mathbf{Z}G_1$ , i.e.

$$D = AX + BY + P\bar{X} + Q\bar{Y},$$

and  $\pi D = 7 \cdot S$  with

$$S = r + \sum_{i=1}^6 s_i U_i + \sum_{j=1}^6 t_j V_j,$$

where  $S = A + B + P + Q$ . Thus, all coefficients  $r, s_1, \dots, s_6, t_1, \dots, t_6$  are non-negative integers  $\leq 4$ .

Again  $\pi(D\bar{D}) = 1225 + 1176 \cdot (1 + \sum U_i + \sum V_j) \cdot 29$ . Therefore,

$$S^2 = 25 + 696 \cdot \left(1 + \sum_{i=1}^6 U_i + \sum_{j=1}^6 V_j\right).$$

With our (abuse of) notation, we set  $G_1 = 1 + \sum U_i + \sum V_j$ . Then,  $G_1^2 = 169 \cdot G_1$ . Thus, we see that

$$S = \pm (5 + 2G_1)$$

are solutions of  $S^2 = 25 + 696 \cdot G_1$ . We claim that there is no other. This will clearly finish the non-existence proof since  $r \leq 4$ . Note the decomposition

$$\mathbf{Q}G_1 = \mathbf{Q} \times \mathbf{Q}(\zeta_{13}) \times \mathbf{Q}(\zeta_{169})$$

of the algebra  $\mathbf{Q}G_1$  as a product of fields, where  $\zeta_{13}$  is a primitive 13-th root of unity, and  $\zeta_{169}$  a primitive 169-th root of unity.

The element  $G_1 = \sum_{k=0}^{168} z^k \in \mathbf{Z}G_1$  corresponds on the right hand side to  $(169, 0, 0)$  since  $\zeta_{13}$  and  $\zeta_{169}$  are roots of the polynomial  $\sum_{k=0}^{168} X^k$ . It follows that  $S^2 = (343^2, 5^2, 5^2)$ . Hence, any solution  $Z \in \mathbf{Z}G_1$  of the equation  $Z^2 = 25 + 696G_1$  must correspond to  $(\pm 343, \pm 5, \pm 5)$ . Changing  $Z$  to  $-Z$ , we can assume  $Z = (343, \pm 5, \pm 5)$ . Now, the diagrams

$$\begin{array}{ccc} \mathbf{Z}G_1 & \rightarrow & \mathbf{Z}[\zeta_{13}] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \rightarrow & \mathbf{F}_{13} \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Z}G_1 & \rightarrow & \mathbf{Z}[\zeta_{169}] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \rightarrow & \mathbf{F}_{13} \end{array}$$

where the right vertical arrows send  $\zeta_{13}$ , resp.  $\zeta_{169}$  to  $1 \in \mathbf{F}_{13}$ , are commutative. Since 5 is not congruent to  $-5$  modulo 13, and 343 maps to  $+5 \in \mathbf{F}_{13}$ , we see that  $Z = (343, 5, 5) = S$ .

(3) *Parameters* ( $v = 13613$ ,  $k = 6724$ ,  $\lambda = 3321$ ), *Table I with*  $t = 82$ .

This case is as simple as case (1). Indeed,  $n = 3403 = 41 \cdot 83$ . Since  $41 \equiv 83^3 \pmod{13613}$ , it follows from the multiplier theorem that if a cyclic difference set  $D$  with parameters  $(13613, 6724, 3321)$  existed, then 41 would be a multiplier, and  $D$  could be taken to be a union of orbits under multiplication by 41 on the cyclic group  $\mathbf{Z}/13613\mathbf{Z}$ .

The order of 41 modulo 13613 is 3403, and beside the one-point orbit  $\{0\}$ , there are 4 orbits  $X, iX, i^2X, i^3X$  each of cardinality 3403, where

$$X = \{1, 41, 1681, \dots, 13281\}$$

and  $i$  is a square root of  $-1 \pmod{13613}$ , e.g.  $i = 165$ . Note that 13613 is a prime number.

However, 6724 is not of the form  $n_0 + 3403n_1$  with  $n_0 = 0$  or 1 and  $0 \leq n_1 \leq 4$ . No difference set can therefore have the above parameters.

(4), (5), (6) *Parameters*  $(v, k, \lambda) = (3^3, 13, 6)$ ,  $(3^5, 121, 60)$  and  $(7^3, 171, 85)$  of Table II, with  $n = 7, 61$  and  $86$  respectively.

More generally, we will consider the case

$$(v, k, \lambda) = \left( p^{2t+1}, \frac{p^{2t+1} - 1}{2}, \frac{p^{2t+1} - 3}{4} \right),$$

where  $p$  is a prime  $\equiv 3 \pmod{4}$ .

We have  $n = k - \lambda = \frac{p^{2t+1} + 1}{4}$ . Let  $l_1, \dots, l_r$  be the primes dividing  $n$ .

The group of multipliers for a putative difference set  $D$  with the above parameters contains the intersection  $M$  in  $(\mathbf{Z}/v\mathbf{Z})^*$  of the subgroups generated by  $l_1, \dots, l_r$ . Since  $(\mathbf{Z}/v\mathbf{Z})^*$  is cyclic,  $M$  is the unique subgroup of  $(\mathbf{Z}/v\mathbf{Z})^*$  whose order is the greatest common divisor of the orders  $q_1, \dots, q_r$  of  $l_1, \dots, l_r$  in  $(\mathbf{Z}/v\mathbf{Z})^*$ . We will now assume that the orders  $q_1, \dots, q_r$  of the prime factors  $l_1, \dots, l_r$  of  $n = k - \lambda$  in  $(\mathbf{Z}/v\mathbf{Z})^*$  are all divisible by  $p^{t+1}$ .

**THEOREM.** *There is no cyclic difference set with parameters*

$$(v, k, \lambda) = \left( p^{2t+1}, \frac{p^{2t+1} - 1}{2}, \frac{p^{2t+1} - 3}{4} \right),$$

where  $p$  is a prime  $\equiv 3 \pmod{4}$ , provided that the orders  $q_1, \dots, q_r$  of the prime factors  $l_1, \dots, l_r$  of  $n = k - \lambda$  in  $(\mathbf{Z}/v\mathbf{Z})^*$  are all divisible by  $p^{t+1}$ .

Note that the hypotheses of the theorem above are satisfied for the three examples we have in mind. (Cases  $n = 7, 61$  and  $86$  in Table II.)

(1)  $n = 7$ :  $p = 3, t = 1$ , and 7 is of order  $3^2$  modulo 27;

(2)  $n = 61$ :  $p = 3, t = 2$ , and 61 is of order  $3^4$  modulo 243;

(3)  $n = 86$ :  $p = 7, t = 1$ , and 2 is of order  $3 \cdot 7^2$  modulo 343, 43 is of order  $7^2$  modulo 343.

As expected, the hypothesis on the orders of the prime factors of  $n$  is not satisfied in general. It fails for instance for  $p = 11, t = 1$ : here  $n = \frac{11^3 + 1}{4} = 333 = 3^2 \cdot 37$  and whereas 37 is of order  $5 \cdot 11^2$  modulo  $11^3$ , 3 is only of order  $5 \cdot 11$  modulo  $11^3$ .

However, failure of the hypothesis seems fairly rare: the next example with  $t = 1$  occurs for  $p = 3511$ . Note that 3511 is special for another reason: it satisfies the congruence  $2^{p-1} \equiv 1 \pmod{p^2}$ , the only other known solution being the famous  $p = 1093$ . Such prime numbers are known in the literature as Wieferich prime numbers.

The behaviour of the orders of the prime factors of  $n = \frac{p^{2t+1} + 1}{4}$  in  $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$  is probably a difficult question.

*Proof of the Theorem.* The hypothesis on the orders  $q_1, \dots, q_r$  means that  $m = 1 + p^t$ , which generates the subgroup of order  $p^{t+1}$  in  $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$ , is contained in all the subgroups  $\langle l_1 \rangle, \dots, \langle l_r \rangle$  of  $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$ , and thus is a multiplier of any candidate difference set  $D \subset \mathbf{Z}/p^{2t+1}\mathbf{Z}$  with the above parameters.

What are the orbits of multiplication by  $m = 1 + p^t$  in the ring  $\mathbf{Z}/p^{2t+1}\mathbf{Z}$ ? If  $a_i = i \cdot p^{t+1}$ , then  $a \cdot m \equiv a \pmod{p^{2t+1}}$ . Hence, there are  $p^t$  fixed points  $a_0 = 0, a_1, \dots, a_{p^t-1}$ .

More generally, if  $a_{i,j} = ip^{t-j+1}$  with  $1 \leq i \leq p^t - 1$  and  $\gcd(i, p) = 1$ ,  $j = 1, \dots, t + 1$ , then  $a_{i,j}$  produces an orbit  $\{a_{i,j}m^v\}_{v=0, \dots, p^j-1}$  of length  $p^j$ . Here, we use the formula

$$(1 + p^t)^{p^s} \equiv 1 + p^{t+s} \pmod{p^{t+s+1}}$$

easily proved (for  $p$  odd) by induction on  $s$ , and which implies that  $m$  has (multiplicative) order  $p^j$  modulo  $p^{t+j}$ .

The orbits  $A_{i,j}$  of  $a_{i,j}$  with  $i \in \mathbf{Z}/p^t\mathbf{Z}$  for  $j = 0$  ( $a_{i,0} = a_i$ ), and  $i \in (\mathbf{Z}/p^t\mathbf{Z})^*$  for  $j = 1, \dots, t + 1$  are easily verified to be disjoint. Together, they sweep out

$$p^t + \sum_{j=1}^{t+1} (p-1)p^{t-1} p^j = p^{2t+1}$$

elements of the group  $\mathbf{Z}/p^{2t+1}\mathbf{Z}$ . Hence,  $A_{i,j}$  with  $i \in \mathbf{Z}/p^t\mathbf{Z}$  for  $j = 0$  ( $a_{i,0} = a_i$ ), and  $i \in (\mathbf{Z}/p^t\mathbf{Z})^*$  for  $j = 1, \dots, t + 1$  is the complete collection of orbits under multiplication by  $m = 1 + p^t$  in  $\mathbf{Z}/p^{2t+1}\mathbf{Z}$ . At this point, it may be more convenient to write the group ring of  $\mathbf{Z}/p^{2t+1}\mathbf{Z}$  as  $\mathbf{Z}[x]/(x^{p^{2t+1}} - 1)$ . Identifying a subset  $A \subset \mathbf{Z}/p^{2t+1}\mathbf{Z}$  with the sum of the corresponding elements  $\sum_{a \in A} a$  in the group ring, the orbits  $A_{i,j}$  can then be written as

$$A_{i,j} = \sum_{v=0}^{p^j-1} x^{ip^{t-j+1}m^v}.$$

If a difference set  $D$  with the above parameters exists, it must be of the form

$$D = \sum_{i \in S_0} x^{ip^{t+1}} + \sum_{j=1}^{t+1} \sum_{i \in S_j} A_{i,j}$$

where  $S_0 \subset \mathbf{Z}/p^t\mathbf{Z}$  and  $S_j \subset (\mathbf{Z}/p^t\mathbf{Z})^*$  for  $j = 1, \dots, t+1$ . Now, let  $\pi: \mathbf{Z}[x]/(x^{p^{2t+1}} - 1) \rightarrow \mathbf{Z}[y]/(y^p - 1)$  be the projection of the group ring of  $\mathbf{Z}/p^{2t+1}\mathbf{Z}$  onto the group ring of the cyclic group of order  $p$ . We have  $\pi(x) = y$  and

$$\begin{aligned} \pi A_{i,j} &= p^i \quad \text{for } j = 0, 1, \dots, t \\ \pi A_{i,t+1} &= p^{t+1} \cdot y^i \quad \text{for } i \in (\mathbf{Z}/p^t\mathbf{Z})^*. \end{aligned}$$

It follows that

$$\pi D = s_0 + ps_1 + \dots + p^t s_t + p^{t+1} \left( \sum_{i \in S_{t+1}} y^i \right),$$

where  $s_j = \text{Card}(S_j)$ .

Let  $N = s_0 + ps_1 + \dots + p^t s_t$  and  $a_\mu = \text{Card}\{i \mid i \in S_{t+1}, i \equiv \mu \pmod{p}\}$ , then

$$\pi D = N + p^{t+1} Y,$$

with  $Y = \sum_{\mu=1}^{p-1} a_\mu y^\mu$ . (Note that  $a_0$  is indeed 0 as  $S_{t+1} \subset (\mathbf{Z}/p^t\mathbf{Z})^*$ .)

Therefore  $\pi(D\bar{D}) = \pi(D)\overline{\pi(D)}$  has the form

$$\pi(D\bar{D}) = N^2 + Np^{t+1} \sum_{\mu=1}^{p-1} a_\mu (y^\mu + y^{-\mu}) + p^{2t+2} Y\bar{Y}.$$

On the other hand the condition for  $D$  being a difference set yields, after applying  $\pi$ ,

$$\pi(D\bar{D}) = \frac{p^{2t+1} + 1}{4} + \frac{p^{2t+1} - 3}{4} p^{2t} \left( \sum_{\mu=0}^{p-1} y^\mu \right).$$

We will reach a contradiction by comparing the constant terms (coefficient of 1 in  $\mathbf{Z}[y]/(y^p - 1)$ ) in the two expressions for  $\pi(D\bar{D})$ :

$$N^2 + p^{2t+2} \sum_{\mu=1}^{p-1} a_\mu^2 = \frac{p^{2t+1} + 1}{4} + \frac{p^{2t+1} - 3}{4} p^{2t}.$$

Note that  $k = \text{Card}(D) = N + p^{t+1} s_{t+1}$ , where  $s_{t+1} = \text{Card}(S_{t+1})$ , and hence  $N = \frac{p^{2t+1} - 1}{2} - p^{t+1} s_{t+1}$ . Substituting this in the above equation,

we get

$$4s_{t+1} \equiv 3p^{t-1}(p - 1) \pmod{p^{t+1}}.$$

Writing  $4s_{t+1} = 3p^{t-1}(p - 1) + z \cdot p^{t+1}$  for  $z \in \mathbf{Z}$ , we observe that  $p \equiv 3 \pmod{4}$  implies  $z \equiv 2 \pmod{4}$ , and so  $2p^{t+1} \leq |z \cdot p^{t+1}|$ . But,  $s_{t+1} = \text{Card}(S_{t+1}) \leq p^{t-1}(p - 1)$ , since  $S_{t+1} \subset (\mathbf{Z}/p^t\mathbf{Z})^*$ . It follows that

$$|z \cdot p^{t+1}| \leq |4s_{t+1} - 3p^{t-1}(p - 1)| \leq 3p^{t-1}(p - 1) < 2p^{t+1} \leq |z \cdot p^{t+1}|.$$

We have reached the desired contradiction, i.e. no cyclic difference set with parameters  $\left(p^{2t+1}, \frac{p^{2t+1} - 1}{2}, \frac{p^{2t+1} - 3}{4}\right)$  exists if the orders of the prime factors of  $n = \frac{p^{2t+1} + 1}{4}$  in  $(\mathbf{Z}/p^{2t+1}\mathbf{Z})^*$  are all divisible by  $p^{t+1}$ .  $\square$

(7) *Parameters* ( $v = 399, k = 199, \lambda = 99$ ), *Table II*. This is the last item in Table II, corresponding to  $n = k - \lambda = 100$ .

Since  $4 = 2^2 \equiv 5^8 \pmod{399}$ , it follows that 4 must be a multiplier of any abelian difference set  $D$  with the above parameters.

Writing  $\mathbf{Z}/399\mathbf{Z}$  as a direct product

$$\mathbf{Z}/399\mathbf{Z} = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z} \times \mathbf{Z}/19\mathbf{Z},$$

and accordingly writing the elements of  $\mathbf{Z}/399\mathbf{Z}$  as triples  $g = (x, y, z)$ ,  $x \in \mathbf{Z}/3\mathbf{Z}$ ,  $y \in \mathbf{Z}/7\mathbf{Z}$ ,  $z \in \mathbf{Z}/19\mathbf{Z}$ , we have the following orbits of the multiplication by 4 in  $\mathbf{Z}/399\mathbf{Z}$ : all monomials  $XYZ$ , with  $X \in \{1, U, \bar{U}\}$ ,  $Y \in \{1, V, \bar{V}\}$ ,  $Z \in \{1, W, \bar{W}\}$ , where

$$1 = \{(0, 0, 0)\}$$

$$U = \{(1, 0, 0)\}$$

$$V = \{(0, 1, 0), (0, -3, 0), (0, 2, 0)\}$$

$$W = \{(0, 0, 1), (0, 0, 4), (0, 0, -3), (0, 0, 7), (0, 0, 9), (0, 0, -2), (0, 0, -8), (0, 0, 6), (0, 0, 5)\},$$

and bar denotes the conjugate, i.e. if  $C \subset \mathbf{Z}/v\mathbf{Z}$ , then  $\bar{C} = \{-g \mid g \in C\}$ .

All orbits, except  $1, U, \bar{U}$  have cardinality divisible by 3. Since  $k = 199 \equiv 1 \pmod{3}$ , any putative difference set  $D$  can be assumed to contain a single one-point orbit  $1, U$  or  $\bar{U}$ . Multiplying  $D$  by  $U$  or  $\bar{U}$  if necessary, we may assume that

$$D = 1 + A \cdot V + B \cdot \bar{V} + P \cdot W + Q \cdot \bar{W},$$

where

$$A = \alpha_0 + \alpha_1 U + \alpha_2 \bar{U}, \quad 0 \leq \alpha_i \leq 1,$$

$$B = \beta_0 + \beta_1 U + \beta_2 \bar{U}, \quad 0 \leq \beta_i \leq 1,$$

and  $P, Q$  are polynomials in  $U, \bar{U}$  and  $V, \bar{V}$ .

We first show that  $A$  and  $B$  must be 0. Let  $a = \alpha_0 + \alpha_1 + \alpha_2$ ,  $b = \beta_0 + \beta_1 + \beta_2$ , and let  $\pi: \mathbf{Z}/399\mathbf{Z} \rightarrow \mathbf{Z}/7\mathbf{Z}$  be the projection on the second factor.

We indulge in various abuses of notation: we write  $\pi$  for the group ring projection as well and denote  $\pi V$  again by  $V$ . Note that  $\pi U = \pi \bar{U} = 1$ ,  $\pi W = \pi \bar{W} = 9$ . Then  $\pi D \equiv 1 + aV + b\bar{V} \pmod{9}$ , a congruence in the group ring of  $\mathbf{Z}/7\mathbf{Z}$ .

Since  $D\bar{D} = 100 + 99 \cdot (1 + U + \bar{U})(1 + V + \bar{V})(1 + W + \bar{W})$ , the equation expressing that  $D$  is a difference set with the required parameters, we have  $D\bar{D} \equiv 1 \pmod{9}$ .

Consequently, using

$$V\bar{V} = 3 + V + \bar{V}, \quad V^2 = V + 2\bar{V}, \quad \bar{V}^2 = 2V + \bar{V},$$

we get, expanding  $\pi(D\bar{D}) = \pi(D)\pi(\bar{D})$ , and after collecting terms,

$$3(a^2 + b^2) + (a + b + a^2 + b^2 + 3ab)(V + \bar{V}) \equiv 0 \pmod{9}.$$

Thus,  $a^2 + b^2 \equiv 0 \pmod{3}$ , and this means  $a \equiv b \equiv 0 \pmod{3}$ . But then  $a^2 + b^2 + 3ab \equiv 0 \pmod{9}$ , and so we must also have

$$a + b \equiv 0 \pmod{9},$$

after looking at the coefficient of  $V + \bar{V}$  in the above congruence.

Since  $0 \leq a \leq 3, 0 \leq b \leq 3$ , this means  $a = b = 0$  and therefore  $A = B = 0$ . Any difference set  $D$  with parameters  $(399, 199, 99)$  can therefore be assumed to have the form

$$D = 1 + P \cdot W + Q \cdot \bar{W}.$$

Plugging  $D = 1 + P \cdot W + Q \cdot \bar{W}$  into the equation

$$D\bar{D} = 100 + 99(1 + U + \bar{U})(1 + V + \bar{V})(1 + W + \bar{W})$$

and using the multiplication table

$$W\bar{W} = 9 + 4(W + \bar{W}), \quad W^2 = 4W + 5\bar{W},$$

we get

$$1 + 9(P\bar{P} + Q\bar{Q}) = 100 + 99(1 + U + \bar{U})(1 + V + \bar{V})$$

$$P + \bar{Q} + 4(P\bar{P} + Q\bar{Q}) + 5\bar{P}Q + 4P\bar{Q} = 99(1 + U + \bar{U})(1 + V + \bar{V}),$$

where

$$P = p_0 + p_1U + p_2\bar{U} + (p_3 + p_4U + p_5\bar{U})V + (p_6 + p_7U + p_8\bar{U})\bar{V}$$

$$Q = q_0 + q_1U + q_2\bar{U} + (q_3 + q_4U + q_5\bar{U})V + (q_6 + q_7U + q_8\bar{U})\bar{V}$$

with  $0 \leq p_i, q_i \leq 1$ , for  $i = 0, \dots, 8$ .

The first equation gives

$$P\bar{P} + Q\bar{Q} = 11 + 11(1 + U + \bar{U})(1 + V + \bar{V}).$$

Substituting in the second equation, we get

$$(*) \quad P + \bar{Q} + 5\bar{P}Q + 4P\bar{Q} = -44 + 55(1 + U + \bar{U})(1 + V + \bar{V}).$$

Since  $U\bar{U} = 1$ ,  $U^2 = \bar{U}$  and  $V\bar{V} = 3 + V + \bar{V}$ ,  $V^2 = V + 2\bar{V}$ , the constant terms in  $\bar{P}Q$  and  $P\bar{Q}$  are equal to  $\sum_{i=0}^2 p_i q_i + 3 \sum_{j=3}^8 p_j q_j = c$ , say. Hence, equating constant terms in the above equation (\*), we must have

$$p_0 + q_0 + 9c = 11.$$

The only solution to this equation with all  $p_i, q_i$  being 0 or 1, is  $p_0 = q_0 = 1$ ,  $p_i = q_i = 0$  for  $i = 1, \dots, 8$ . This means  $P = Q = 1$ , contradicting (\*).

## 5. COMMENTS ON THE EXAMPLES IN TABLES II

Difference sets with parameters  $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$  are usually called *Hadamard difference sets*. Our purpose here is to discuss the classification of these cyclic difference sets for  $2 \leq n \leq 100$ .

In many cases where  $v = 4n - 1$  is a prime  $p$ , the quadratic residue difference set, which we denote by  $QR(p)$  is unique for the given values of the parameters. This is obviously the case if the multiplier  $m$  has order  $k = \frac{1}{2}(v - 1)$  in  $(\mathbf{Z}/v\mathbf{Z})^*$ . Indeed, in this case, there are exactly 3 orbits of

multiplication by  $m$  in  $\mathbf{Z}/v\mathbf{Z}$ , namely  $1 = \{0\}$ ,  $M = \{1, m, m^2, \dots, m^{k-1}\}$  and  $\bar{M} = \{-1, -m, \dots, -m^{k-1}\}$ . Thus the only choice for  $D$  is  $D = M$  or  $D = \bar{M}$ , which are isomorphic under conjugation  $\sigma: \mathbf{Z}/v\mathbf{Z} \rightarrow \mathbf{Z}/v\mathbf{Z}$ ,  $\sigma(a) = -a$ .

In our Table II, this situation happens for  $n = 3, 5, 6, 12, 15, 17, 18, 20, 21, 27, 33, 35, 41, 42, 45, 48, 53, 57, 60, 63, 66, 68, 77, 87, 90$  and  $96$ .

The remaining cases where  $v = 4n - 1$  is a prime  $p$  (for  $2 \leq n \leq 100$ ) have been shown to lead to a single difference set, namely  $QR(p)$ , by machine enumeration of the various choices of  $D$  as a union of orbits under multiplication by a multiplier  $m$ . This includes the cases  $n = 26$  (multiplier 8),  $n = 38$  (multiplier 19),  $n = 50$  (multiplier 5),  $n = 78$  (multiplier 13),  $n = 83$  (multiplier 83), and  $n = 95$  (multiplier 5). By far, the most difficult case (for the machine) occurs with  $n = 38$ , which required the examination of 37 442 160 combinations of multiplier orbits.