Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 38 (1992)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE CATEGORY OF NILMANIFOLDS
Autor: Oprea, John
Kapitel: §3. NILMANIFOLDS
DOI: https://doi.org/10.5169/seals-59480

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 21.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

In order to show the reverse inequality $s \leqslant k$, we must show that, for $p: \Lambda X \to \Lambda X/\Lambda^{>k}X, p^*$ is injective. Plainly, by Poincaré duality, p^* is injective if and only if $p^*(\tau) \neq 0$. Hence, we prove this.

Suppose $p^*(\tau) = 0$. Let τ denote the representing cocycle in $\Lambda^{\geqslant k}X$ of the fundamental class τ. Let $p(\tau) = \bar{\tau} \in \Lambda X/\Lambda^{>k}X$ and consider $\bar{\tau}$ as an element in $\Lambda^{\leqslant k}X$. Now, $p^*(\tau) = 0$, so there exists $\bar{\alpha} \in \Lambda X/\Lambda^{>k}X$ with $\bar{d}\bar{\alpha} = \bar{\tau}$. Consider $\bar{\alpha} \in \Lambda^{\leqslant k}X$ as well and note that $p(d\bar{\alpha}) = \bar{d}\bar{\alpha} = \bar{\tau}$. Therefore, in ΛX

$$d\bar{\alpha} = \bar{\tau} + \Phi\,, \qquad \text{where } \Phi \in \Lambda^{>k}X\,.$$

Similarly, of course, $\tau = \bar{\tau} + \Omega$ for $\Omega \in \Lambda^{>k}X$ and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with $\Omega - \Phi \in \Lambda^{>k}X$. But this means τ is cohomologous to $\Omega - \Phi \in \Lambda^{>k}X$, contradicting the definition of k. $\square$

§3. NILMANIFOLDS

A *nilmanifold* M is the quotient of a nilpotent Lie group N by a discrete cocompact subgroup π. The description below follows [7].

It is well known that N is diffeomorphic to some $\mathbf{R}^n$ and, therefore, M is a $K(\pi, 1)$. Furthermore, this entails the fact that π is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of π,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each $\pi_i/\pi_{i+1} \cong \mathbf{Z}$ whose length is invariant and is called the *rank* of π. So, for π above, $\operatorname{rank}(\pi) = n$.

This description implies that any $u \in \pi$ has a decomposition $u = u_1^{x_1} \cdots u_n^{x_n}$, where $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i/\pi_{i+1}$. The set $\{u_1, \cdots u_n\}$ is called a Malcev basis for π. Using this basis the multiplication in π takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x,y)} \cdots u_n^{\rho_n(x,y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})\,.$$

Example. $N = U_n(\mathbf{R})$, the group of upper diagonal matrices with 1's on the diagonal; $\pi = U_n(\mathbf{Z})$. A Malcev basis is given by $\{u_{ij} \mid 1 \leqslant i < j \leqslant n\}$ where $u_{ij} = I + e_{ij}$ and

$$\rho_{ij}(x, y) = x_{ij} + y_{ij} + \sum_{i<k<j} x_{ik}y_{kj} .$$

Consider the central extension $\pi_n \to \pi \to \bar{\pi}$. The cocycle for the extension is $\tau_n\colon \bar{\pi} \times \bar{\pi} \to \mathbf{Z}$. Of course $\bar{\pi}$ is also finitely generated torsionfree with refined upper central series,

$$\bar{\pi} = \frac{\pi}{\pi_n} \supseteq \frac{\pi_2}{\pi_n} \supseteq \cdots \supseteq \frac{\pi_{n-1}}{\pi_n} \supseteq \frac{\pi_n}{\pi_n} = 1 .$$

Hence, $\operatorname{rank}(\bar{\pi}) = n - 1$ and

$$\bar{\rho}_i(x, y) = \rho_i((x, 0), (y, 0)) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})$$

for $i < n$. Clearly, then, we may iterate this process and decompose π as n central extensions of the form

$$\mathbf{Z} \to G \to \bar{G}$$

with cocycles $\tau_i \in H^2(\bar{G}; \mathbf{Z})$ (with untwisted coefficients since the extension is central).

This desription allows a geometric formulation:

$$\tau_n \in H^2(\bar{\pi}; \mathbf{Z}) \cong H^2(K(\bar{\pi}, 1); \mathbf{Z}) \cong [K(\bar{\pi}, 1), K(\mathbf{Z}, 2)]$$

by the usual identification of cohomology groups with sets of homotopy classes into $K(\mathbf{Z}, m)$'s. Now, $K(\mathbf{Z}, 2) = \mathbf{C}P(\infty)$, the classifying space for principal S^1-bundles, so τ_n induces a bundle over $K(\bar{\pi}, 1)$,

$$\begin{array}{ccc} S^1 & \to & K(\pi, 1) \\ & & \downarrow \\ & & K(\bar{\pi}, 1) \xrightarrow{\tau_n} \mathbf{C}P(\infty) . \end{array}$$

The total space of the bundle is clearly $K(\pi, 1)$ since the ensuing short exact sequence of fundamental groups is classified by τ_n.

Now, because we can iterate the algebraic decomposition of π, we obtain an iterated sequence of principal S^1-bundles classifed by the τ_i:

$$\begin{array}{ccccc}
S^1 & \to & M & = & K(\pi, 1) \\
 & & \downarrow & & \\
S^1 & \to & M_{n-1} & \xrightarrow{\tau_n} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
S^1 & \to & M_1 & \xrightarrow{\tau_2} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & * & \xrightarrow{\tau_1} & \mathbf{C}P(\infty)\ .
\end{array}$$

We can assume (by finite dimensionality) that each τ_i has image in a finite $\mathbf{C}P(n)$, so thus may be approximated by a smooth map. Hence, each M_j is a compact manifold with

$$\dim(M_j) = \dim(M_{j-1}) + 1\ .$$

Thus, $\dim(M) = \operatorname{rank}(\pi) = n$.

§4. Category of nilmanifolds

The decomposition of $M = K(\pi, 1)$ into a tower of principal S^1-bundles is, in fact, the Postnikov decomposition of M with k-invariants the τ_i. By the fundamental theorem of rational homotopy theory, the minimal model has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_n), d)\ , \qquad \deg(x_i) = 1$$

with $dx_i = \tau_i$, where τ_i is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbf{Z})$. Note that $\Lambda(M)$ is an exterior algebra because all generators are in degree 1. Therefore, since $\dim M = n$, the only possibility for a cocycle representing the fundamental class is $x_1 \cdots x_n$. Hence, $e_0(M) = n$ and this immediately implies,

Proof of Theorem 1. $n = e_0(M) \leqslant \operatorname{cat}_0(M) \leqslant \operatorname{cat}(M) \leqslant \dim M = n$. $\square$

Example. Consider the 3-dimensional Heisenberg group $U_3(\mathbf{R})$ and mod out by $U_3(\mathbf{Z})$. The resulting M is a 3-manifold obtained as a principal bundle,

$$S^1 \to M \to T^2$$