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This follows from the obvious congruence  $c_j \equiv l - j \mod 2$ , and the fact that  $c_j \in \{-1, 0, +1\}$ , for all j = 1, ..., l - 1.

Now, applying the relation  $ab \equiv a + b - 1 \mod 4$  for any  $a, b = \pm 1$ , we have

(2) 
$$c_j = \sum_{i=1}^{l-j} a_i a_{i+j} \equiv \sum_{i=1}^{l-j} (a_i + a_{i+j}) - (l-j) \mod 4$$

for j = 1, ..., l - 1.

Comparing the above congruences for two successive values of j, we obtain

(3) 
$$c_j - c_{j+1} \equiv a_{l-j} + a_{j+1} - 1 \mod 4$$
,

for j = 1, ..., l - 2.

Changing j to l - j - 1 leaves the right-hand-side unchanged. Therefore, we have

(4) 
$$c_j - c_{j+1} \equiv c_{l-j-1} - c_{l-j} \mod 4$$
,

for j = 1, ..., l - 2. Since  $|c_j - c_{j+1}| \leq 1$  for all j by (1), we have in fact an equality:

$$c_j - c_{j+1} = c_{l-j-1} - c_{l-j}$$

for j = 1, ..., l - 2. Using Lemma 1, it follows that

$$\gamma_j = \gamma_{j+1}$$

for all j = 1, ..., l - 2, and thus  $\gamma_j$  is independent of j, as claimed.

Now  $|\gamma_j| = |c_j + c_{l-j}| \leq 2$ , and equality can occur only if  $c_j = c_{l-j} = \pm 1$ , which by (1) implies in particular that *j* must be odd. But this is impossible, because  $\gamma_j$  is independent of *j*. Therefore  $|\gamma_j| \leq 1$ , as claimed.

## 1. DIFFERENCE SETS

In this section, we show that the notion of a binary sequence with constant periodic correlations is equivalent to that of a difference set on a cyclic group. We then recall basic results concerning these difference sets.

Definition. A difference set D on a group G is a subset  $D \subset G$  such that the cardinality of the intersection

$$D \cap g \cdot D$$

is independent of g for  $g \in G \setminus \{e\}$ . Here,  $gD = \{gx \mid x \in D\}$  is the translate of D by the element  $g \in G$ , and e is the neutral element of G.

It is traditional to denote by v the cardinality of G, by k the cardinality of D and by  $\lambda$  the cardinality of the intersection  $D \cap gD$ :

$$v = |G|, \quad k = |D|, \quad \lambda = |D \cap gD|.$$

The difference set D in G is then said to have *parameters*  $(v, k, \lambda)$ . It is also traditional to denote by n the difference  $k - \lambda$ .

Observe that if  $D \in G$  is a difference set, then so is  $D' = G \setminus D$ . Thus we can and will always assume that  $k = |D| \leq \frac{1}{2}v$ .

Note that if  $D \in G$  is a difference set, the collection of *right translates* of D, including D itself, viz.

$$\mathscr{B} = \{ Dg \mid g \in G \}$$

constitutes a symmetric block design on G. This means that each element of G is contained in exactly k blocks (recall k = |D|), and every pair of (distinct) elements of G belongs to precisely  $\lambda$  blocks.

Indeed, if  $g \in G$ , let  $g_x = x^{-1}g$ ; then

$$g \in Dg_x$$
 if and only if  $x \in D$ 

and therefore the correspondence  $x \mapsto Dg_x$  provides a bijection between D and the set of blocks containing g.

If  $g_1, g_2 \in G$  is a pair of distinct elements of G, set  $g_x = x^{-1}g_1$ . Then,

$$g_1, g_2 \in Dg_x$$
 if and only if  $x \in D \cap g_1g_2^{-1}D$ 

and the assignment  $x \mapsto Dg_x$  establishes a bijection between  $D \cap g_1g_2^{-1}D$ of cardinality  $\lambda$  and the set of blocks Dg containing the pair  $g_1, g_2$ .

PROPOSITION. There is a bijection between the set of binary sequences  $A = (a_1, ..., a_v)$  with constant periodic correlation  $\gamma$ , i.e.

$$\gamma = \sum_{i \bmod v} a_i \cdot a_{i+j}$$

for j = 1, ..., v - 1, and difference sets D on the cyclic group  $G = \mathbb{Z}/v\mathbb{Z}$ of order v with parameters  $(v, k, \lambda)$ , where  $\lambda = k - (v - \gamma)/4$ . The set D associated to the sequence A is given by  $D = \{i \mid a_i = -1\}$ .

*Remark.* In particular, if there is a binary sequence of length v with constant periodic correlation  $\gamma$ , then one must have  $v \equiv \gamma \mod 4$ , and  $\gamma$  is given by

$$\gamma = v - 4n$$

where, as above,  $n = k - \lambda$ .

We call  $\gamma = v - 4n$  the *correlation* of the cyclic difference set D with parameters  $(v, k, \lambda)$ .

In the proposition we must momentarily relax our convention  $|D| \leq |G|/2$ .

**Proof.** Let  $G = \mathbb{Z}/v\mathbb{Z}$ . We will represent the elements of G by  $\{1, 2, ..., v\}$ . Suppose  $A = (a_1, ..., a_v)$  is a binary sequence and  $\gamma = \sum_{i=1}^{v} a_i a_{i+j}$  is independent of j for j = 1, ..., v - 1. To A we associate the subset

$$D = \{i \mid a_i = -1\} \subset G \; .$$

Set k = |D|. We claim that

$$\lambda = |D \cap (j+D)| = k - (v-\gamma)/4$$

for all  $j \neq 0$ . Indeed, we have

$$\gamma = \sum_{i=1}^{r} a_{i}a_{i+j} = |D' \cap (j+D')| + |D \cap (j+D)| - |D \cap (j+D')|$$
$$- |D' \cap (j+D)|,$$

where  $D' = G \setminus D$ .

Now, we have

(1) 
$$|D \cap (j+D)| + |D \cap (j+D')| = k$$

(2) 
$$|D \cap (j+D)| + |D' \cap (j+D)| = k$$

(3) 
$$|D' \cap (j + D')| + |D \cap (j + D')| = v - k$$

(4)  $|D' \cap (j + D')| + |D' \cap (j + D)| = v - k$ 

from which we conclude (by comparing (1) and (2)):

$$|D \cap (j+D')| = |D' \cap (j+D)| = k - \lambda$$

and (by substracting (3) from (1)):

$$|D \cap (j+D)| - |D' \cap (j+D')| = 2k - v$$
.

Comparing this with

$$\gamma = \left| D \cap (j+D) \right| + \left| D' \cap (j+D') \right| - 2(k-\lambda) ,$$

we get the desired relation

$$2\lambda = 2k - v + \gamma + 2(k - \lambda) .$$

Conversely, if  $D \in \mathbb{Z}/v\mathbb{Z}$  is a cyclic difference set, then viewing D as a subset of  $\{1, ..., v\}$ , define  $a_i = +1$  if  $i \notin D$  and  $a_i = -1$  if  $i \in D$ . The periodic correlations  $\gamma = \sum_{i \mod v} a_i a_{i+j}$  (j = 1, ..., v - 1) are independent of j and have the common value  $\gamma = v - 4n$ .

Equivalently, we may recast the proof as follows: write

$$D(z) = \sum_{d \in D} z^d \in \mathbb{Z}[z]/(z^v - 1)$$

if  $D \in \mathbb{Z}/v\mathbb{Z}$ . We see that D is a difference set with parameters  $(v, k, \lambda)$  if and only if

(1) 
$$D(z)D(z^{-1}) = n + \lambda T,$$

where  $n = k - \lambda$  and  $T = 1 + z + \cdots + z^{\nu-1}$ . Now,  $A(z) = \sum_{i=1}^{\nu} a_i z^{i-1}$  has constant periodic correlation  $\gamma$  if and only if

(2) 
$$A(z)A(z^{-1}) = v + \gamma(T-1)$$
 in  $\mathbf{Z}[z]/(z^{v}-1)$ 

If  $D \in \mathbb{Z}/v\mathbb{Z}$  is the set of exponents of the monomials  $z^i$  occurring with coefficient -1 in A(z), then A(z) = T - 2D(z), where  $D(z) = \sum_{d \in D} z^d$  as above.

An easy calculation, using  $T(z^{-1}) = T(z)$  and  $z \cdot T(z) = T(z)$ , shows that (2) is equivalent to

$$D(z)D(z^{-1}) = \frac{v-\gamma}{4} + \left(k-\frac{v-\gamma}{4}\right) T$$

and therefore (2) is equivalent to *D* being a cyclic difference set with parameters  $(v, k, \lambda)$ , where  $\lambda = k - \frac{v - \gamma}{4}$ .

Note that a difference set on a group G could equivalently be defined as a subset D of a G-set E such that

(1) |E| = |G|,

(2) G acts transitively on E, i.e. E affords the regular representation of G, and

(3)  $\lambda = |D \cap gD|$  is independent of g for  $g \in G \setminus \{1\}$ .

We shall sometimes use this presentation in the sequel.

Several necessary conditions must be satisfied by a given triple  $(v, k, \lambda)$  to be realized as the parameters of some difference set. These well known conditions are recalled below. We refer to [L] for more details.

First of all, the triple  $(v, k, \lambda)$  must satisfy the equation

$$k(k-1) = \lambda(v-1) .$$

This follows easily from the definition of a symmetric block design. Next, we have:

- (1) if v is even, then  $n = k \lambda$  must be a square (Schützenberger);
- (2) if v is odd, the equation

$$nX^{2} + (-1)^{\frac{1}{2}(v-1)} \lambda Y^{2} = Z^{2}$$

must have a solution  $(X, Y, Z) \neq (0, 0, 0)$  in integers (Chowla-Ryser).

A deeper condition on the parameters of a difference set in an *abelian* group is provided by the following result. First we need a

Definition. A prime number p is said to be semi-primitive modulo the positive integer w if there is some integer f for which the equation

$$p^f \equiv -1 \mod w$$

holds. A number m is said to be *semi-primitive* modulo w if all its prime factors are. Finally, the number m is said to be *self-conjugate* modulo w, if m is semi-primitive modulo w', where w' denotes the largest divisor of w which is prime to m.

SEMI-PRIMITIVITY THEOREM. Suppose that there exists a  $(v, k, \lambda)$ difference set in an abelian group G. Let p be any prime divisor of  $n = k - \lambda$ . Then p is not semi-primitive modulo the exponent e(G)of G.

Furthermore, if p divides the square-free part of n, then there is no divisor w > 1 of v = |G| for which p is semi-primitive mod w.

(See [L], Theorem 4.5, page 134.)

Another very useful theorem of R. Turyn is:

TURYN'S INEQUALITY. Assume a non-trivial  $(v, k, \lambda)$  difference set in a cyclic group exists. Let m > 1 be an integer such that  $m^2$  divides  $n = k - \lambda$  and such that m is self-conjugate modulo w for some divisor w > 1 of v. If gcd(m, w) = 1 then  $m \leq v/w$ . If gcd(m, w) > 1 then

$$m \leqslant 2^{r-1} v/w ,$$

where r is the number of distinct prime factors of gcd(m, w).

(See [T1]; in the special case r = 1, see also [Y] and [R].)

We now turn to one of the *multiplier theorems*, which sometimes describes a difference set as a union of orbits under multiplication by a certain integer. First a Definition. Let G be a finite abelian group and D a difference set on G. The integer m is a multiplier for D if m is prime to v = |G|, and if the isomorphism  $m: G \to G$  induced by multiplication by m, permutes the translates a + D ( $a \in G$ ) of D.

Thus, m is a multiplier if (m, v) = 1, and if  $m \cdot D = a + D$  for some  $a \in G$ .

We will also need the following result:

PROPOSITION. Let *m* be a multiplier of a difference set *D* in an abelian group *G*. Then some translate D' = a + D ( $a \in G$ ) of *D*, is fixed under multiplication by *m*, i.e.  $m \cdot D' = D'$ .

This follows at once from a more general result, stating that an automorphism of a symmetric block design fixes as many points as blocks. (See [L], Theorem 3.1, page 78.) In our context, the multiplication by m in Gfixes 0, hence it must fix at least one translate of D.

As a consequence, if an abelian difference set D admits a multiplier m, we may very well suppose that D is fixed under multiplication by m, and thus, that D is a union of orbits under multiplication by m.

The multiplier theorem below tells us how to find multipliers of abelian difference sets.

MULTIPLIER THEOREM. Let D be a  $(v, k, \lambda)$  difference set in an abelian group G. Let  $n_1$  be a divisor of  $n = k - \lambda$  such that  $n_1 > \lambda$ . Suppose m is an integer satisfying

(1) gcd(m, v) = 1;

(2) for every prime divisor p of  $n_1$ , m is a power of p modulo the exponent e of G.

Then, m is a multiplier of the difference set D.

In Section 4, we will use this theorem to exclude the existence of periodic Barker sequences of various lengths.

# 2. PERIODIC BARKER SEQUENCES

This section deals with periodic Barker sequences, i.e. binary sequences whose periodic correlations  $\gamma_j$  are constant and equal to  $\gamma \in \{0, 1, -1\}$ .

Case  $\gamma = 0$ . In this case, the parameters  $(v, k, \lambda)$  and  $n = k - \lambda$  of the associated cyclic difference set (see Section 1) satisfy:

 $n = N^2$ ,  $v = 4N^2$ ,  $k = 2N^2 - N$ ,  $\lambda = N^2 - N$ .