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This follows from the obvious congruence  $c_j \equiv l - j \pmod{2}$ , and the fact that  $c_j \in \{-1, 0, +1\}$ , for all  $j = 1, \dots, l - 1$ .

Now, applying the relation  $ab \equiv a + b - 1 \pmod{4}$  for any  $a, b = \pm 1$ , we have

$$(2) \quad c_j = \sum_{i=1}^{l-j} a_i a_{i+j} \equiv \sum_{i=1}^{l-j} (a_i + a_{i+j}) - (l-j) \pmod{4}$$

for  $j = 1, \dots, l - 1$ .

Comparing the above congruences for two successive values of  $j$ , we obtain

$$(3) \quad c_j - c_{j+1} \equiv a_{l-j} + a_{j+1} - 1 \pmod{4},$$

for  $j = 1, \dots, l - 2$ .

Changing  $j$  to  $l - j - 1$  leaves the right-hand-side unchanged. Therefore, we have

$$(4) \quad c_j - c_{j+1} \equiv c_{l-j-1} - c_{l-j} \pmod{4},$$

for  $j = 1, \dots, l - 2$ . Since  $|c_j - c_{j+1}| \leq 1$  for all  $j$  by (1), we have in fact an equality:

$$c_j - c_{j+1} = c_{l-j-1} - c_{l-j}$$

for  $j = 1, \dots, l - 2$ . Using Lemma 1, it follows that

$$\gamma_j = \gamma_{j+1}$$

for all  $j = 1, \dots, l - 2$ , and thus  $\gamma_j$  is independent of  $j$ , as claimed.

Now  $|\gamma_j| = |c_j + c_{l-j}| \leq 2$ , and equality can occur only if  $c_j = c_{l-j} = \pm 1$ , which by (1) implies in particular that  $j$  must be odd. But this is impossible, because  $\gamma_j$  is independent of  $j$ . Therefore  $|\gamma_j| \leq 1$ , as claimed.  $\square$

## 1. DIFFERENCE SETS

In this section, we show that the notion of a binary sequence with constant periodic correlations is equivalent to that of a difference set on a cyclic group. We then recall basic results concerning these difference sets.

*Definition.* A difference set  $D$  on a group  $G$  is a subset  $D \subset G$  such that the cardinality of the intersection

$$D \cap g \cdot D$$

is independent of  $g$  for  $g \in G \setminus \{e\}$ . Here,  $gD = \{gx \mid x \in D\}$  is the translate of  $D$  by the element  $g \in G$ , and  $e$  is the neutral element of  $G$ .

It is traditional to denote by  $v$  the cardinality of  $G$ , by  $k$  the cardinality of  $D$  and by  $\lambda$  the cardinality of the intersection  $D \cap gD$ :

$$v = |G|, \quad k = |D|, \quad \lambda = |D \cap gD|.$$

The difference set  $D$  in  $G$  is then said to have *parameters*  $(v, k, \lambda)$ . It is also traditional to denote by  $n$  the difference  $k - \lambda$ .

Observe that if  $D \subset G$  is a difference set, then so is  $D' = G \setminus D$ . Thus we can and will always assume that  $k = |D| \leq \frac{1}{2}v$ .

Note that if  $D \subset G$  is a difference set, the collection of *right translates* of  $D$ , including  $D$  itself, viz.

$$\mathcal{B} = \{Dg \mid g \in G\}$$

constitutes a *symmetric block design* on  $G$ . This means that each element of  $G$  is contained in exactly  $k$  blocks (recall  $k = |D|$ ), and every pair of (distinct) elements of  $G$  belongs to precisely  $\lambda$  blocks.

Indeed, if  $g \in G$ , let  $g_x = x^{-1}g$ ; then

$$g \in Dg_x \quad \text{if and only if} \quad x \in D$$

and therefore the correspondence  $x \mapsto Dg_x$  provides a bijection between  $D$  and the set of blocks containing  $g$ .

If  $g_1, g_2 \in G$  is a pair of distinct elements of  $G$ , set  $g_x = x^{-1}g_1$ . Then,

$$g_1, g_2 \in Dg_x \quad \text{if and only if} \quad x \in D \cap g_1g_2^{-1}D$$

and the assignment  $x \mapsto Dg_x$  establishes a bijection between  $D \cap g_1g_2^{-1}D$  of cardinality  $\lambda$  and the set of blocks  $Dg$  containing the pair  $g_1, g_2$ .

**PROPOSITION.** *There is a bijection between the set of binary sequences  $A = (a_1, \dots, a_v)$  with constant periodic correlation  $\gamma$ , i.e.*

$$\gamma = \sum_{i \bmod v} a_i \cdot a_{i+j}$$

for  $j = 1, \dots, v - 1$ , and difference sets  $D$  on the cyclic group  $G = \mathbf{Z}/v\mathbf{Z}$  of order  $v$  with parameters  $(v, k, \lambda)$ , where  $\lambda = k - (v - \gamma)/4$ . The set  $D$  associated to the sequence  $A$  is given by  $D = \{i \mid a_i = -1\}$ .

*Remark.* In particular, if there is a binary sequence of length  $v$  with constant periodic correlation  $\gamma$ , then one must have  $v \equiv \gamma \pmod{4}$ , and  $\gamma$  is given by

$$\gamma = v - 4n,$$

where, as above,  $n = k - \lambda$ .

We call  $\gamma = v - 4n$  the *correlation* of the cyclic difference set  $D$  with parameters  $(v, k, \lambda)$ .

In the proposition we must momentarily relax our convention  $|D| \leq |G|/2$ .

*Proof.* Let  $G = \mathbf{Z}/v\mathbf{Z}$ . We will represent the elements of  $G$  by  $\{1, 2, \dots, v\}$ . Suppose  $A = (a_1, \dots, a_v)$  is a binary sequence and  $\gamma = \sum_{i=1}^v a_i a_{i+j}$  is independent of  $j$  for  $j = 1, \dots, v-1$ . To  $A$  we associate the subset

$$D = \{i \mid a_i = -1\} \subset G.$$

Set  $k = |D|$ . We claim that

$$\lambda = |D \cap (j + D)| = k - (v - \gamma)/4$$

for all  $j \neq 0$ . Indeed, we have

$$\begin{aligned} \gamma = \sum_{i=1}^v a_i a_{i+j} &= |D' \cap (j + D')| + |D \cap (j + D)| - |D \cap (j + D')| \\ &\quad - |D' \cap (j + D)|, \end{aligned}$$

where  $D' = G \setminus D$ .

Now, we have

- (1)  $|D \cap (j + D)| + |D \cap (j + D')| = k$
- (2)  $|D \cap (j + D)| + |D' \cap (j + D)| = k$
- (3)  $|D' \cap (j + D')| + |D \cap (j + D')| = v - k$
- (4)  $|D' \cap (j + D')| + |D' \cap (j + D)| = v - k$

from which we conclude (by comparing (1) and (2)):

$$|D \cap (j + D')| = |D' \cap (j + D)| = k - \lambda$$

and (by subtracting (3) from (1)):

$$|D \cap (j + D)| - |D' \cap (j + D')| = 2k - v.$$

Comparing this with

$$\gamma = |D \cap (j + D)| + |D' \cap (j + D')| - 2(k - \lambda),$$

we get the desired relation

$$2\lambda = 2k - v + \gamma + 2(k - \lambda).$$

Conversely, if  $D \subset \mathbf{Z}/v\mathbf{Z}$  is a cyclic difference set, then viewing  $D$  as a subset of  $\{1, \dots, v\}$ , define  $a_i = +1$  if  $i \notin D$  and  $a_i = -1$  if  $i \in D$ . The periodic correlations  $\gamma = \sum_{i \bmod v} a_i a_{i+j}$  ( $j = 1, \dots, v-1$ ) are independent of  $j$  and have the common value  $\gamma = v - 4n$ .

Equivalently, we may recast the proof as follows: write

$$D(z) = \sum_{d \in D} z^d \in \mathbf{Z}[z]/(z^v - 1)$$

if  $D \subset \mathbf{Z}/v\mathbf{Z}$ . We see that  $D$  is a difference set with parameters  $(v, k, \lambda)$  if and only if

$$(1) \quad D(z)D(z^{-1}) = n + \lambda T,$$

where  $n = k - \lambda$  and  $T = 1 + z + \cdots + z^{v-1}$ . Now,  $A(z) = \sum_{i=1}^v a_i z^{i-1}$  has constant periodic correlation  $\gamma$  if and only if

$$(2) \quad A(z)A(z^{-1}) = v + \gamma(T - 1) \quad \text{in} \quad \mathbf{Z}[z]/(z^v - 1)$$

If  $D \subset \mathbf{Z}/v\mathbf{Z}$  is the set of exponents of the monomials  $z^i$  occurring with coefficient  $-1$  in  $A(z)$ , then  $A(z) = T - 2D(z)$ , where  $D(z) = \sum_{d \in D} z^d$  as above.

An easy calculation, using  $T(z^{-1}) = T(z)$  and  $z \cdot T(z) = T(z)$ , shows that (2) is equivalent to

$$D(z)D(z^{-1}) = \frac{v - \gamma}{4} + \left( k - \frac{v - \gamma}{4} \right) T$$

and therefore (2) is equivalent to  $D$  being a cyclic difference set with parameters

$$(v, k, \lambda), \quad \text{where} \quad \lambda = k - \frac{v - \gamma}{4}. \quad \square$$

Note that a difference set on a group  $G$  could equivalently be defined as a subset  $D$  of a  $G$ -set  $E$  such that

$$(1) \quad |E| = |G|,$$

(2)  $G$  acts transitively on  $E$ , i.e.  $E$  affords the regular representation of  $G$ , and

$$(3) \quad \lambda = |D \cap gD| \text{ is independent of } g \text{ for } g \in G \setminus \{1\}.$$

We shall sometimes use this presentation in the sequel.

Several necessary conditions must be satisfied by a given triple  $(v, k, \lambda)$  to be realized as the parameters of some difference set. These well known conditions are recalled below. We refer to [L] for more details.

First of all, the triple  $(v, k, \lambda)$  must satisfy the equation

$$k(k - 1) = \lambda(v - 1).$$

This follows easily from the definition of a symmetric block design. Next, we have:

- (1) if  $v$  is even, then  $n = k - \lambda$  must be a square (Schützenberger);
- (2) if  $v$  is odd, the equation

$$nX^2 + (-1)^{\frac{1}{2}(v-1)} \lambda Y^2 = Z^2$$

must have a solution  $(X, Y, Z) \neq (0, 0, 0)$  in integers (Chowla-Ryser).

A deeper condition on the parameters of a difference set in an *abelian group* is provided by the following result. First we need a

*Definition.* A prime number  $p$  is said to be *semi-primitive* modulo the positive integer  $w$  if there is some integer  $f$  for which the equation

$$p^f \equiv -1 \pmod{w}$$

holds. A number  $m$  is said to be *semi-primitive* modulo  $w$  if all its prime factors are. Finally, the number  $m$  is said to be *self-conjugate* modulo  $w$ , if  $m$  is semi-primitive modulo  $w'$ , where  $w'$  denotes the largest divisor of  $w$  which is prime to  $m$ .

**SEMI-PRIMITIVITY THEOREM.** *Suppose that there exists a  $(v, k, \lambda)$ -difference set in an abelian group  $G$ . Let  $p$  be any prime divisor of  $n = k - \lambda$ . Then  $p$  is not semi-primitive modulo the exponent  $e(G)$  of  $G$ .*

*Furthermore, if  $p$  divides the square-free part of  $n$ , then there is no divisor  $w > 1$  of  $v = |G|$  for which  $p$  is semi-primitive mod  $w$ .*

(See [L], Theorem 4.5, page 134.)

Another very useful theorem of R. Turyn is:

**TURYN'S INEQUALITY.** *Assume a non-trivial  $(v, k, \lambda)$  difference set in a cyclic group exists. Let  $m > 1$  be an integer such that  $m^2$  divides  $n = k - \lambda$  and such that  $m$  is self-conjugate modulo  $w$  for some divisor  $w > 1$  of  $v$ . If  $\gcd(m, w) = 1$  then  $m \leq v/w$ . If  $\gcd(m, w) > 1$  then*

$$m \leq 2^{r-1} v/w,$$

*where  $r$  is the number of distinct prime factors of  $\gcd(m, w)$ .*

(See [T1]; in the special case  $r = 1$ , see also [Y] and [R].)

We now turn to one of the *multiplier theorems*, which sometimes describes a difference set as a union of orbits under multiplication by a certain integer. First a

*Definition.* Let  $G$  be a finite abelian group and  $D$  a difference set on  $G$ . The integer  $m$  is a *multiplier* for  $D$  if  $m$  is prime to  $v = |G|$ , and if the isomorphism  $m: G \rightarrow G$  induced by multiplication by  $m$ , permutes the translates  $a + D$  ( $a \in G$ ) of  $D$ .

Thus,  $m$  is a multiplier if  $(m, v) = 1$ , and if  $m \cdot D = a + D$  for some  $a \in G$ .

We will also need the following result:

**PROPOSITION.** *Let  $m$  be a multiplier of a difference set  $D$  in an abelian group  $G$ . Then some translate  $D' = a + D$  ( $a \in G$ ) of  $D$ , is fixed under multiplication by  $m$ , i.e.  $m \cdot D' = D'$ .*

This follows at once from a more general result, stating that an automorphism of a symmetric block design fixes as many points as blocks. (See [L], Theorem 3.1, page 78.) In our context, the multiplication by  $m$  in  $G$  fixes 0, hence it must fix at least one translate of  $D$ .

As a consequence, if an abelian difference set  $D$  admits a multiplier  $m$ , we may very well suppose that  $D$  is fixed under multiplication by  $m$ , and thus, that  $D$  is a *union of orbits under multiplication by  $m$* .

The multiplier theorem below tells us how to find multipliers of abelian difference sets.

**MULTIPLIER THEOREM.** *Let  $D$  be a  $(v, k, \lambda)$  difference set in an abelian group  $G$ . Let  $n_1$  be a divisor of  $n = k - \lambda$  such that  $n_1 > \lambda$ . Suppose  $m$  is an integer satisfying*

$$(1) \quad \gcd(m, v) = 1;$$

(2) *for every prime divisor  $p$  of  $n_1$ ,  $m$  is a power of  $p$  modulo the exponent  $e$  of  $G$ .*

*Then,  $m$  is a multiplier of the difference set  $D$ .*

In Section 4, we will use this theorem to exclude the existence of periodic Barker sequences of various lengths.

## 2. PERIODIC BARKER SEQUENCES

This section deals with periodic Barker sequences, i.e. binary sequences whose periodic correlations  $\gamma_j$  are constant and equal to  $\gamma \in \{0, 1, -1\}$ .

*Case  $\gamma = 0$ .* In this case, the parameters  $(v, k, \lambda)$  and  $n = k - \lambda$  of the associated cyclic difference set (see Section 1) satisfy:

$$n = N^2, \quad v = 4N^2, \quad k = 2N^2 - N, \quad \lambda = N^2 - N.$$