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Artikel: RATIONALITY OF PIECEWISE LINEAR FOLIATIONS AND HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR HOMEOMORPHISMS

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Kapitel: §5. SURJECTIVITY OF $(j_+)_\star$

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is equivalent to the independence of the cohomology classes of these cocycles $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$. To show the independence we use the following theorem.

THEOREM (4.1). *Let* $j_+ : PL_c([0, \infty)) \to \mathbf{R}$ *denote the homomorphism defined by*

$$j_+(f) = \log f'(0)\,.$$

The homomorphism j_+ *induces a surjection in integer homology.*

Using this theorem, we can show the independence. Let $u_i^- \otimes_{\mathbf{Q}} u_i^+$ be an element of $V^{k_i^-, k_i^+}$ $(u_i^- \in \mathbf{R}^{\wedge k_i^-}, u_i^+ \in \mathbf{R}^{\wedge k_i^+})$. Then we have a k_i^--dimensional cycle σ_i^- of $BPL_c((-\infty, 0])^\delta$ such that the image under $(j_-)_*$ coincides with $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^\delta; \mathbf{Z})$, where $j_- : PL_c((-\infty, 0]) \to \mathbf{R}$ denotes the homomorphism defined by $j_-(f) = \log f'(0)$. We also have a k_i^+-dimensional cycle σ_i^+ of $BPL_c([0, \infty))^\delta$ such that the image under $(j_+)_*$ coincides with $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^\delta; \mathbf{Z})$. Then $\sigma_i^- \times \sigma_i^+$ is a $(k_i^- + k_i^+)$-dimensional cycle of $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^\delta$ such that the image under $(j_- \times j_+)_*$ coincides with $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$. Now let $T_1, \ldots, T_s$ be translations of $\mathbf{R}$ such that $T_1(0) < \ldots < T_s(0)$ and the supports of $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+) T_i^{-1}$ are contained in disjoint open intervals, where the support of a cycle of $BPL_c(\mathbf{R})^\delta$ is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then $\sigma_1 \times \ldots \times \sigma_s$ is an m-cycle and the value of the cocycle $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ on it is $(u_1^- \otimes_{\mathbf{Q}} u_1^+) \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} (u_s^- \otimes_{\mathbf{Q}} u_s^+)$. It is easy to see that the values of the other m-cocycles on this cycle are 0.

The fact that $*$-product coincides with the tensor product follows from Lemma (1.2). Note that the map s in Lemma (1.2) is an isomorphism from the subgroup of $H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ generated by the $\sigma^- \times \sigma^+$ to $H_{*+1}(B\overline{\Gamma}_1^{PL}; \mathbf{Z})$. Thus Theorem (3.1) is proved.

§5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider j_+ as a homomorphism from $PL_c([0, \infty))$ to the group of germs at 0. We use the fact that the n-dimensional homology group of $B\mathbf{R}^\delta$ is isomorphic to $\mathbf{R}^{\wedge n}$ and whose generators are represented by the images of the fundamental classes of tori T^n of dimension n under the mappings which are defined by n (commuting) elements. We will construct an n-complex Y_n with the fundamental class and a degree one map $Y_n \to T^n$. Then for each mapping $T^n \to B\mathbf{R}^\delta$, we will construct a mapping $Y_n \to BPL_c([0, \infty))^\delta$ such that the following diagram commutes.

$$\begin{array}{ccc} Y_n & \to & BPL_c([0, \infty))^\delta \\ \downarrow & & \downarrow \\ T^n & \to & B\mathbf{R}^\delta . \end{array}$$

Theorem (4.1) follows immediately from this commutative diagram.

Construction of Y_n. Let L be a large positive real number. In the Euclidean n space, we consider the following polyhedron X_n

$$X_n = \{(x_1, \ldots, x_n) \in [0, L]^n ; \quad x_{i_1} + \ldots + x_{i_k} \geqslant (k-1)k/2 \\ \text{for} \quad 1 \leqslant i_1 < \ldots < i_k \leqslant n\} .$$

The shape of X_n is the cube with certain neighborhoods of the k-faces ($k \leqslant n-2$) in the coordinate planes deleted, those of the $(k-1)$-faces being thicker than those of the k-faces.

The polyhedron X_n has $2^n - 1 + n$ faces of dimension $n-1$. If $(x_1, \ldots, x_n)$ is a vertex of X_n then $(x_1, \ldots, x_n)$ is a permutation of $(0, 1, .., k, L, \ldots, L)$. In this case we say $(x_1, \ldots, x_n)$ is a vertex of type $\{0, 1, .., k, L, \ldots, L\}$. There are edges between $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ of the same type $\{0, 1, .., k, L, \ldots, L\}$ if one is obtained from the other by permuting two coordinates. The edges between different types exists only if the types are $\{0, 1, .., k-1, L, \ldots, L\}$ and $\{0, 1, .., k, L, \ldots, L\}$, and one vertex is obtained from the other by changing the entries k and L.

The polyhedron X_n has the $(n-1)$-face $\{x_i = L\}$ which is isometric to X_{n-1}. The $(n-1)$-face $\{x_i = 0\}$ is isometric to X_{n-1} with L replaces by $L-1$ because if $x_i = 0$ then

$$x_{i_1} + \ldots + x_{i_k} \geqslant (k-1)k/2$$

for $\{i_1, \ldots, i_k\}$ containing i implies

$$(x_{i_1} - 1) + \ldots + (x_{i_k} - 1) \geqslant (k-1)k/2$$

for $\{i_1, \ldots, i_k\}$ not containing i. Hence we can define a simplicial identification between the faces $\{x_i = L\}$ and $\{x_i = 0\}$. In general, the face

$$\{x_{i_1} + \ldots + x_{i_k} = (k-1)k/2\}$$

is isometric to $X'_{n-k} \times \Sigma_k$, where X'_{n-k} is X_{n-k} with L replaced by $L-k$ and Σ_k is the face $\{x_1 + \ldots + x_k = (k-1)k/2\}$ in X_k. The reason is

$$x_{i'_1} + \ldots + x_{i'_{k'}} \geqslant (k'-1)k'/2$$

for $\{i'_1, \ldots, i'_{k'}\}$ containing $\{i_1, \ldots, i_k\}$ implies

$$(x_{i'_1} - k) + \ldots + (x_{i'_{k'}} - k) \geqslant (k'-1)k'/2$$

for $\{i'_1, \ldots, i'_{k'}\}$ not containing $\{i_1, \ldots, i_k\}$. We also fix a simplicial identification between X'_{n-k} and X_{n-k}. Now we distinguish the faces by the set $\{i_1, \ldots, i_k\}$ of indices and we see that

$$\partial X_n = \bigcup_{A \subset \{1,\ldots,n\},\, \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times \Sigma_A$$
$$\cup \bigcup_i X^{(L)}_{\{1,\ldots,n\}-\{i\}} \cup \bigcup_i X^{(0)}_{\{1,\ldots,n\}-\{i\}} ,$$

where

$$X_{\{1,\ldots,n\}-A} \times \Sigma_A = \{x_{i_1} + \ldots + x_{i_k} = (k-1)k/2\} \quad \text{if} \quad A = \{i_1, \ldots, i_k\} ,$$
$$X^{(L)}_{\{1,\ldots,n\}-\{i\}} = \{x_i = L\} \quad \text{and} \quad X^{(0)}_{\{1,\ldots,n\}-\{i\}} = \{x_i = 0\} .$$

The complex Y_n is defined inductively as follows. $Y_1 = X_1 = [0, L]$. Y_2 is obtained from X_2 (a pentagon) by identifying $X^{(L)}_{\{i\}}$ and $X^{(0)}_{\{i\}}$ $(i = 1, 2)$ and by taking the double of it. Hence Y_2 is a surface of genus 2. We call the new part in the double $B\Sigma_{\{1,2\}}$.

$$Y_2 = X_2 + B\Sigma_{\{1,2\}} .$$

Y_3 is obtained from X_3 by identifying $X^{(L)}_{\{i,j\}}$ and $X^{(0)}_{\{i,j\}}$ $(i, j = 1, 2, 3)$, by attaching $X_{\{k\}} \times B\Sigma_{\{i,j\}}$ $(\{i, j, k\} = \{1, 2, 3\})$ to each $X_{\{k\}} \times \Sigma_{\{i,j\}}$, and then by taking the double. The boundary before taking the double is a surface of genus 6. We call the new part in the double $B\Sigma_{\{1,2,3\}}$.

$$Y_3 = X_3 + \sum_{\{i_1, i_2\} \subset \{1,2,3\}} X_{\{1,2,3\}-\{i_1,i_2\}} \times B\Sigma_{\{i_1,i_2\}} + B\Sigma_{\{1,2,3\}} .$$

In general, we define Y_n to be the double of

$$X_n + \sum_{A \subset \{1,\ldots,n\},\, \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times B\Sigma_A$$

and we call the new part in the double $B\Sigma_{\{1,\ldots,n\}}$.

$$Y_n = X_n + \sum_{A \subset \{1,\ldots,n\},\, \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times B\Sigma_A + B\Sigma_{\{1,\ldots,n\}} .$$

The mapping from Y_n to T^n is the one which sends the all $B\Sigma_A$ parts to a point and X_n to the fundamental domain of T^n.

Construction of $Y_n \to BPL_c([0,\infty))^\delta$. Now given a mapping $T^n \to B\mathbf{R}^\delta$, we construct a mapping $Y_n \to BPL_c([0,\infty))^\delta$. In other words, given a homomorphism $\mathbf{Z}^n \to \mathbf{R}$, we construct a homomorphism $\pi_1(Y_n) \to PL_c([0,\infty))$. This is also done inductively.

For $n = 1$, it is only necessary to choose a lift in $PL_c([0,\infty))$ of an element of $\mathbf{R}$.

Now for $n = 2$, we choose lifts f_1, f_2 of the generators of $\mathbf{Z}^2$. To the edges of Y_2, we associate elements of $PL_c([0,\infty))$. We put f_1 on the edges of X_2 from (L,L) to $(0,L)$ and from $(L,0)$ to $(1,0)$, and we put f_2 on the edges of X_2 from (L,L) to $(L,0)$ and from $(0,L)$ to $(0,1)$. Then we put the commutator $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$ on the edge from $(0,1)$ to $(1,0)$. Note that the support of this commutator does not contain 0 hence this commutator is an element of $PL_c((0,\infty))$. This commutator is also written as a commutator of elements of $PL_c((0,\infty))$. We can do it very easily, not by using the perfectness of the group $PL_c((0,\infty))$, but by using a conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a(>0)$ and which is the identity on $(2a,\infty)$ when the support of $[f_1, f_2]$ is contained in $(2a,\infty)$. We call this conjugation c_*. (This technique using conjugation is similar to that in [12].) c_* is an isomorphism from $PL_c([0,\infty))$ to a subgroup of $PL_c((0,\infty))$. Then $[f_1, f_2] = c_*([f_1, f_2]) = [c_* f_1, c_* f_2]$ and we associate $c_* f_1, c_* f_2$ to the edges in the new part in the double (in the mirror). Thus we defined the desired mapping $Y_2 \to BPL_c([0,\infty))^\delta$.

For general n, we use the same strategy. First we choose lifts $f_1, \ldots, f_n$ of the generators of $\mathbf{Z}^n$. To the edges of X_n, we associate elements of $PL_c([0,\infty))$. We associate f_i to the edge from a vertex of type $\{0,1,..,k-1,L,\ldots,L\}$ to a vertex of type $\{0,1,..,k,L,\ldots,L\}$ if the i-th coordinate changes from L to k. Then the elements associated to other edges are uniquely determined. In fact, we can associate an element of $PL_c([0,\infty))$ to each vertices as follows. We associate id to the vertex of type $\{L,\ldots,L\}$, if we already associated an element f_υ to a vertex υ of type $\{0,1,..,k-1,L,\ldots,L\}$ and a vertex υ' is obtained from υ by changing the i-th coordinate from L to k then we associate $f_i f_\upsilon$ to the vertex υ'. Thus the edge from one vertex υ_1 to another vertex υ_2 is associated with $f_{\upsilon_2} f_{\upsilon_1}^{-1}$. Now if we look at the edges of Σ_A in the $(n-1)$-face $X_{\{1,\ldots,n\}-A} \times \Sigma_A$ the associated elements are in $PL_c((0,\infty))$. By induction, we can find $B\Sigma_A$ with edges in $PL_c((0,\infty))$. Thus we find the boundary of

$$X_n + \sum_{A \subset \{1,\ldots,n\},\ \# A \geqslant 2} X_{\{1,\ldots,n\}-A} \times B\Sigma_A$$

is a cycle of $PL_c((0, \infty))$. Here the products are considered as in the following remark. Hence in the double Y_n, we can associate the images under c_* in the new part of the double. (c_* is the conjugation by an element of $PL_c(\mathbf{R})$ which sends 0 to $a'(>0)$ and which is the identity on $(2a', \infty)$ when the support of the above boundary is contained in $(2a', \infty)$.) Thus we defined the desired mapping $Y_n \to BPL_c([0, \infty))^\delta$. This proves Theorem (4.1).

Remark. For two simplices $(g_1, \ldots, g_m)$ and $(h_{m+1}, \ldots, h_{m+n})$ of the classifying space for a discrete group, we define the product of them as follows.

$$(g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n}) = \sum_\sigma \operatorname{sign}(\sigma)\, (f_{\sigma,1}, \ldots, f_{\sigma,m+n}) \,.$$

where the sum is taken over the shuffles σ (that is, those permutations such that $\sigma(1) < \ldots < \sigma(m)$ and $\sigma(m+1) < \ldots < \sigma(m+n)$. The entry $f_{\sigma,j}$ is defined as follows.

$$f_{\sigma,\sigma(j)} = g_j \quad (j = 1, \ldots, m) \quad \text{and}$$
$$f_{\sigma,m+j} = (g_k \cdots g_m) h_{m+j} (g_k \cdots g_m)^{-1} \quad (j = 1, \ldots, n)\,,$$

where k is the integer such that $\sigma(k-1) < \sigma(m+j) < \sigma(k)$. For example, $(g_1, g_2) \times (h_3, h_4)$

$$\begin{aligned}
&= (g_1, g_2, h_3, h_4) - (g_1, g_2 h_3 g_2^{-1}, g_2, h_4)\\
&\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2, h_4) + (g_1, g_2 h_3 g_2^{-1}, g_2 h_4 g_2^{-1}, g_2)\\
&\quad - (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2 h_4 g_2^{-1}, g_2)\\
&\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1 g_2 h_4 (g_1 g_2)^{-1}, g_1, g_2).
\end{aligned}$$

This product is defined so that

$$\begin{aligned}
&\partial((g_1, \ldots, g_m) \times (h_{m+1}, \ldots, h_{m+n}))\\
&= (\partial'(g_1, \ldots, g_m)) \times (h_{m+1}, \ldots, h_{m+n})\\
&\quad + (-1)^m (g_1, \ldots, g_{m-1}) \times (g_m h_{m+1} g_m^{-1}, \ldots, g_m h_{m+n} g_m^{-1})\\
&\quad + (-1)^m (g_1, \ldots, g_m) \times (\partial(h_{m+1}, \ldots, h_{m+n}))\,,
\end{aligned}$$

where

$$\begin{aligned}
\partial(g_1, \ldots, g_m) &= (g_2, \ldots, g_m)\\
+ \sum_{i=1}^{m-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_m) &+ (-1)^m (g_1, \ldots, g_{m-1})\\
&= \partial'(g_1, \ldots, g_m) + (-1)^m (g_1, \ldots, g_{m-1})\,.
\end{aligned}$$

For the above complex we triangulate it and associate the elements for their products.