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is equivalent to the independence of the cohomology classes of these cocycles  $C^{m}_{(k_{1}^{-}, k_{1}^{+}, ..., k_{s}^{-}, k_{s}^{+})}$ . To show the independence we use the following theorem.

THEOREM (4.1). Let  $j_+: PL_c([0, \infty)) \to \mathbb{R}$  denote the homomorphism defined by

$$j_+(f) = \log f'(0) \; .$$

The homomorphism  $j_+$  induces a surjection in integer homology.

Using this theorem, we can show the independence. Let  $u_i^- \otimes_{\mathbf{Q}} u_i^+$  be an element of  $V^{k_i^-,k_i^+}(u_i^- \in \mathbb{R}^{\wedge k_i^-}, u_i^+ \in \mathbb{R}^{\wedge k_i^+})$ . Then we have a  $k_i^-$ -dimensional cycle  $\sigma_i^-$  of  $BPL_c((-\infty, 0])^{\delta}$  such that the image under  $(j_-)_*$  coincides with  $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^{\delta}; \mathbf{Z})$ , where  $j_-: PL_c((-\infty, 0]) \to \mathbf{R}$  denotes the homomorphism defined by  $j_{-}(f) = \log f'(0)$ . We also have а  $k_i^+$ -dimensional cycle  $\sigma_i^+$  of  $BPL_c([0, \infty))^{\delta}$  such that the image under  $(j_+)_*$ coincides with  $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^{\delta}; \mathbf{Z})$ . Then  $\sigma_i^- \times \sigma_i^+$  is a  $(k_i^- + k_i^+)$ -dimensional cycle of  $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^{\delta}$  such that the image under  $(j_- \times j_+)_*$  coincides with  $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$ . Now let  $T_1, ..., T_s$  be translations of **R** such that  $T_1(0) < ... < T_s(0)$  and the supports of  $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+) T_i^{-1}$  are contained in disjoint open intervals, where the support of a cycle of  $BPL_c(\mathbf{R})^{\delta}$  is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then  $\sigma_1 \times \ldots \times \sigma_s$  is an *m*-cycle and the value of the cocycle  $C^{m}_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$  on it is  $(u_1^- \otimes_Q u_1^+) \otimes_Q \dots \otimes_Q (u_s^- \otimes_Q u_s^+)$ . It is easy to see that the values of the other *m*-cocycles on this cycle are 0.

The fact that \*-product coincides with the tensor product follows from Lemma (1.2). Note that the map s in Lemma (1.2) is an isomorphism from the subgroup of  $H_*(BPL_c(\mathbf{R})^{\delta}; \mathbf{Z})$  generated by the  $\sigma^- \times \sigma^+$  to  $H_{*+1}(B\Gamma_1^{PL}; \mathbf{Z})$ . Thus Theorem (3.1) is proved.

# §5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider  $j_+$  as a homomorphism from  $PL_c([0, \infty))$  to the group of germs at 0. We use the fact that the *n*-dimensional homology group of  $B\mathbf{R}^{\delta}$  is isomorphic to  $\mathbf{R}^{\wedge n}$  and whose generators are represented by the images of the fundamental classes of tori  $T^n$  of dimension *n* under the mappings which are defined by *n* (commuting) elements. We will construct an *n*-complex  $Y_n$  with the fundamental class and a degree one

map  $Y_n \to T^n$ . Then for each mapping  $T^n \to B\mathbf{R}^{\delta}$ , we will construct a mapping  $Y_n \to BPL_c([0, \infty))^{\delta}$  such that the following diagram commutes.

$$Y_n \rightarrow BPL_c([0,\infty))^{\delta}$$

$$\downarrow \qquad \qquad \downarrow$$

$$T^n \rightarrow BR^{\delta}$$

Theorem (4.1) follows immediately from this commutative diagram.

Construction of  $Y_n$ . Let L be a large positive real number. In the Euclidean n space, we consider the following polyhedron  $X_n$ 

$$X_n = \{ (x_1, ..., x_n) \in [0, L]^n ; \quad x_{i_1} + ... + x_{i_k} \ge (k-1)k/2$$
  
for  $1 \le i_1 < ... < i_k \le n \}$ .

The shape of  $X_n$  is the cube with certain neighborhoods of the k-faces  $(k \le n-2)$  in the coordinate planes deleted, those of the (k-1)-faces being thicker than those of the k-faces.

The polyhedron  $X_n$  has  $2^n - 1 + n$  faces of dimension n - 1. If  $(x_1, ..., x_n)$  is a vertex of  $X_n$  then  $(x_1, ..., x_n)$  is a permutation of (0, 1, ..., k, L, ..., L). In this case we say  $(x_1, ..., x_n)$  is a vertex of type  $\{0, 1, ..., k, L, ..., L\}$ . There are edges between  $(x_1, ..., x_n)$  and  $(x'_1, ..., x'_n)$  of the same type  $\{0, 1, ..., k, L, ..., L\}$  if one is obtained from the other by permuting two coordinates. The edges between different types exists only if the types are  $\{0, 1, ..., k - 1, L, ..., L\}$  and  $\{0, 1, ..., k, L, ..., L\}$ , and one vertex is obtained from the other by changing the entries k and L.

The polyhedron  $X_n$  has the (n-1)-face  $\{x_i = L\}$  which is isometric to  $X_{n-1}$ . The (n-1)-face  $\{x_i = 0\}$  is isometric to  $X_{n-1}$  with L replaces by L - 1 because if  $x_i = 0$  then

$$x_{i_1} + \ldots + x_{i_k} \ge (k-1)k/2$$

for  $\{i_1, ..., i_k\}$  containing *i* implies

$$(x_{i_1} - 1) + \dots + (x_{i_k} - 1) \ge (k - 1)k/2$$

for  $\{i_1, ..., i_k\}$  not containing *i*. Hence we can define a simplicial identification between the faces  $\{x_i = L\}$  and  $\{x_i = 0\}$ . In general, the face

$${x_{i_1} + \ldots + x_{i_k} = (k-1)k/2}$$

is isometric to  $X'_{n-k} \times \Sigma_k$ , where  $X'_{n-k}$  is  $X_{n-k}$  with L replaced by L - kand  $\Sigma_k$  is the face  $\{x_1 + \ldots + x_k = (k-1)k/2\}$  in  $X_k$ . The reason is

$$x_{i'_1} + \dots + x_{i'_{k'}} \ge (k'-1)k'/2$$

for  $\{i'_1, ..., i'_{k'}\}$  containing  $\{i_1, ..., i_k\}$  implies

$$(x_{i'_1} - k) + \dots + (x_{i'_{k'}} - k) \ge (k' - 1)k'/2$$

for  $\{i'_1, ..., i'_{k'}\}$  not containing  $\{i_1, ..., i_k\}$ . We also fix a simplicial identification between  $X'_{n-k}$  and  $X_{n-k}$ . Now we distinguish the faces by the set  $\{i_1, ..., i_k\}$  of indices and we see that

$$\partial X_n = \bigcup_{A \in \{1, ..., n\}, \ \# \ A \ge 2} X_{\{1, ..., n\} - A} \times \Sigma_A$$
  
 $\cup \bigcup_i X^{(L)}_{\{1, ..., n\} - \{i\}} \cup \bigcup_i X^{(0)}_{\{1, ..., n\} - \{i\}},$ 

where

$$X_{\{1,...,n\}-A} \times \Sigma_A = \{x_{i_1} + ... + x_{i_k} = (k-1)k/2\} \text{ if } A = \{i_1,...,i_k\}, \\ X_{\{1,...,n\}-\{i\}}^{(L)} = \{x_i = L\} \text{ and } X_{\{1,...,n\}-\{i\}}^{(0)} = \{x_i = 0\}.$$

The complex  $Y_n$  is defined inductively as follows.  $Y_1 = X_1 = [0, L]$ .  $Y_2$  is obtained from  $X_2$  (a pentagon) by identifying  $X_{\{i\}}^{(L)}$  and  $X_{\{i\}}^{(0)}$  (i = 1, 2) and by taking the double of it. Hence  $Y_2$  is a surface of genus 2. We call the new part in the double  $B\Sigma_{\{1,2\}}$ .

$$Y_2 = X_2 + B\Sigma_{\{1,2\}} .$$

 $Y_3$  is obtained from  $X_3$  by identifying  $X_{\{i,j\}}^{(L)}$  and  $X_{\{i,j\}}^{(0)}(i, j = 1, 2, 3)$ , by attaching  $X_{\{k\}} \times B\Sigma_{\{i,j\}}(\{i, j, k\} = \{1, 2, 3\})$  to each  $X_{\{k\}} \times \Sigma_{\{i,j\}}$ , and then by taking the double. The boundary before taking the double is a surface of genus 6. We call the new part in the double  $B\Sigma_{\{1,2,3\}}$ .

$$Y_3 = X_3 + \sum_{\{i_1, i_2\} \in \{1, 2, 3\}} X_{\{1, 2, 3\} - \{i_1, i_2\}} \times B\Sigma_{\{i_1, i_2\}} + B\Sigma_{\{1, 2, 3\}}.$$

In general, we define  $Y_n$  to be the double of

$$X_n + \sum_{A \in \{1, ..., n\}, \ \# A \ge 2} X_{\{1, ..., n\} - A} \times B\Sigma_A$$

and we call the new part in the double  $B\Sigma_{\{1,...,n\}}$ .

$$Y_n = X_n + \sum_{A \in \{1, ..., n\}, \# A \ge 2} X_{\{1, ..., n\} - A} \times B\Sigma_A + B\Sigma_{\{1, ..., n\}}.$$

The mapping from  $Y_n$  to  $T^n$  is the one which sends the all  $B\Sigma_A$  parts to a point and  $X_n$  to the fundamental domain of  $T^n$ .

Construction of  $Y_n \to BPL_c([0, \infty))^{\delta}$ . Now given a mapping  $T^n \to B\mathbf{R}^{\delta}$ , we construct a mapping  $Y_n \to BPL_c([0, \infty))^{\delta}$ . In other words, given a homomorphism  $\mathbf{Z}^n \to \mathbf{R}$ , we construct a homomorphism  $\pi_1(Y_n)$  $\to PL_c([0, \infty))$ . This is also done inductively.

For n = 1, it is only necessary to choose a lift in  $PL_c([0, \infty))$  of an element of **R**.

Now for n = 2, we choose lifts  $f_1$ ,  $f_2$  of the generators of  $\mathbb{Z}^2$ . To the edges of  $Y_2$ , we associate elements of  $PL_c([0, \infty))$ . We put  $f_1$  on the edges of  $X_2$  from (L, L) to (0, L) and from (L, 0) to (1, 0), and we put  $f_2$  on the edges of  $X_2$  from (L, L) to (L, 0) and from (0, L) to (0, 1). Then we put the commutator  $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$  on the edge from (0, 1) to (1, 0). Note that the support of this commutator does not contain 0 hence this commutator is an element of  $PL_c((0, \infty))$ . This commutator is also written as a commutator of elements of  $PL_c((0,\infty))$ . We can do it very easily, not by using the perfectness of the group  $PL_c((0,\infty))$ , but by using a conjugation by an element of  $PL_c(\mathbf{R})$  which sends 0 to a(>0) and which is the identity on  $(2a, \infty)$  when the support of  $[f_1, f_2]$  is contained in  $(2a, \infty)$ . We call this conjugation  $c_*$ . (This technique using conjugation is similar to that in [12].)  $c_*$  is an isomorphism from  $PL_c([0,\infty))$  to a subgroup of  $PL_c((0,\infty))$ . Then  $[f_1, f_2] = c_*([f_1, f_2]) = [c_*f_1, c_*f_2]$  and we associate  $c_*f_1, c_*f_2$  to the edges in the new part in the double (in the mirror). Thus we defined the desired mapping  $Y_2 \to BPL_c([0,\infty))^{\delta}$ .

For general n, we use the same strategy. First we choose lifts  $f_1, ..., f_n$  of the generators of  $\mathbb{Z}^n$ . To the edges of  $X_n$ , we associate elements of  $PL_c([0,\infty))$ . We associate  $f_i$  to the edge from a vertex of type  $\{0, 1, .., k - 1, L, ..., L\}$  to a vertex of type  $\{0, 1, ..., k, L, ..., L\}$  if the *i*-th coordinate changes from L to k. Then the elements associated to other edges are uniquely determined. In fact, we can associate an element of  $PL_c([0,\infty))$ to each vertices as follows. We associate id to the vertex of type  $\{L, ..., L\}$ , already associated an element  $f_v$  to a vertex v of type if we  $\{0, 1, .., k-1, L, ..., L\}$  and a vertex v' is obtained from v by changing the *i*-th coordinate from L to k then we associate  $f_i f_v$  to the vertex v'. Thus the edge from one vertex  $v_1$  to another vertex  $v_2$  is associated with  $f_{v_2} f_{v_1}^{-1}$ . Now if we look at the edges of  $\Sigma_A$  in the (n-1)-face  $X_{\{1,...,n\}-A} \times \Sigma_A$  the associated elements are in  $PL_c((0, \infty))$ . By induction, we can find  $B\Sigma_A$  with edges in  $PL_c((0, \infty))$ . Thus we find the boundary of

 $X_n + \sum_{A \in \{1, ..., n\}, \# A \ge 2} X_{\{1, ..., n\} - A} \times B\Sigma_A$ 

is a cycle of  $PL_c((0, \infty))$ . Here the products are considered as in the following remark. Hence in the double  $Y_n$ , we can associate the images under  $c_*$  in the new part of the double. ( $c_*$  is the conjugation by an element of  $PL_c(\mathbb{R})$  which sends 0 to a'(>0) and which is the identity on  $(2a', \infty)$  when the support of the above boundary is contained in  $(2a', \infty)$ .) Thus we defined the desired mapping  $Y_n \to BPL_c([0, \infty))^{\delta}$ . This proves Theorem (4.1).

*Remark.* For two simplices  $(g_1, ..., g_m)$  and  $(h_{m+1}, ..., h_{m+n})$  of the classifying space for a discrete group, we define the product of them as follows.

$$(g_1, ..., g_m) \times (h_{m+1}, ..., h_{m+n}) = \sum_{\sigma} \operatorname{sign}(\sigma) (f_{\sigma, 1}, ..., f_{\sigma, m+n})$$

where the sum is taken over the shuffles  $\sigma$  (that is, those permutations such that  $\sigma(1) < ... < \sigma(m)$  and  $\sigma(m+1) < ... < \sigma(m+n)$ . The entry  $f_{\sigma,j}$  is defined as follows.

$$f_{\sigma,\sigma(j)} = g_j$$
  $(j = 1,...,m)$  and  
 $f_{\sigma,m+j} = (g_k...g_m)h_{m+j}(g_k...g_m)^{-1}$   $(j = 1,...,n)$ ,

where k is the integer such that  $\sigma(k-1) < \sigma(m+j) < \sigma(k)$ . For example,  $(g_1, g_2) \times (h_3, h_4)$ 

$$= (g_1, g_2, h_3, h_4) - (g_1, g_2 h_3 g_2^{-1}, g_2, h_4) + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2, h_4) + (g_1, g_2 h_3 g_2^{-1}, g_2 h_4 g_2^{-1}, g_2) - (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2 h_4 g_2^{-1}, g_2) + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1 g_2 h_4 (g_1 g_2)^{-1}, g_1, g_2).$$

This product is defined so that

$$\begin{aligned} \partial \big( (g_1, \, \dots, \, g_m) \, \times \, (h_{m+1}, \, \dots, \, h_{m+n}) \big) \\ &= \big( \partial'(g_1, \, \dots, \, g_m) \big) \, \times \, (h_{m+1}, \, \dots, \, h_{m+n}) \\ &+ \, (-1)^m(g_1, \, \dots, \, g_{m-1}) \, \times \, (g_m h_{m+1} g_m^{-1}, \, \dots, \, g_m h_{m+n} g_m^{-1}) \\ &+ \, (-1)^m(g_1, \, \dots, \, g_m) \, \times \, \big( \partial (h_{m+1}, \, \dots, \, h_{m+n}) \big) , \end{aligned}$$

where

$$\partial(g_1, ..., g_m) = (g_2, ..., g_m) + \sum_{i=1}^{m-1} (-1)^i (g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_m) + (-1)^m (g_1, ..., g_{m-1}) = \partial'(g_1, ..., g_m) + (-1)^m (g_1, ..., g_{m-1}) .$$

For the above complex we triangulate it and associate the elements for their products.