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HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR  
HOMEOMORPHISMS  
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is equivalent to the independence of the cohomology classes of these cocycles  $C_{(k_1^-, k_1^+, \dots, k_s^-, k_s^+)}^m$ . To show the independence we use the following theorem.

**THEOREM (4.1).** *Let  $j_+ : PL_c([0, \infty)) \rightarrow \mathbf{R}$  denote the homomorphism defined by*

$$j_+(f) = \log f'(0).$$

*The homomorphism  $j_+$  induces a surjection in integer homology.*

Using this theorem, we can show the independence. Let  $u_i^- \otimes_{\mathbf{Q}} u_i^+$  be an element of  $V^{k_i^-, k_i^+}$  ( $u_i^- \in \mathbf{R}^{\wedge k_i^-}$ ,  $u_i^+ \in \mathbf{R}^{\wedge k_i^+}$ ). Then we have a  $k_i^-$ -dimensional cycle  $\sigma_i^-$  of  $BPL_c((-\infty, 0])^\delta$  such that the image under  $(j_-)_*$  coincides with  $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^\delta; \mathbf{Z})$ , where  $j_- : PL_c((-\infty, 0]) \rightarrow \mathbf{R}$  denotes the homomorphism defined by  $j_-(f) = \log f'(0)$ . We also have a  $k_i^+$ -dimensional cycle  $\sigma_i^+$  of  $BPL_c([0, \infty))^\delta$  such that the image under  $(j_+)_*$  coincides with  $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^\delta; \mathbf{Z})$ . Then  $\sigma_i^- \times \sigma_i^+$  is a  $(k_i^- + k_i^+)$ -dimensional cycle of  $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^\delta$  such that the image under  $(j_- \times j_+)_*$  coincides with  $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$ . Now let  $T_1, \dots, T_s$  be translations of  $\mathbf{R}$  such that  $T_1(0) < \dots < T_s(0)$  and the supports of  $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+)T_i^{-1}$  are contained in disjoint open intervals, where the support of a cycle of  $BPL_c(\mathbf{R})^\delta$  is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then  $\sigma_1 \times \dots \times \sigma_s$  is an  $m$ -cycle and the value of the cocycle  $C_{(k_1^-, k_1^+, \dots, k_s^-, k_s^+)}^m$  on it is  $(u_1^- \otimes_{\mathbf{Q}} u_1^+) \otimes_{\mathbf{Q}} \dots \otimes_{\mathbf{Q}} (u_s^- \otimes_{\mathbf{Q}} u_s^+)$ . It is easy to see that the values of the other  $m$ -cocycles on this cycle are 0.

The fact that  $*$ -product coincides with the tensor product follows from Lemma (1.2). Note that the map  $s$  in Lemma (1.2) is an isomorphism from the subgroup of  $H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$  generated by the  $\sigma^- \times \sigma^+$  to  $H_{*+1}(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$ . Thus Theorem (3.1) is proved.

## §5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider  $j_+$  as a homomorphism from  $PL_c([0, \infty))$  to the group of germs at 0. We use the fact that the  $n$ -dimensional homology group of  $B\mathbf{R}^\delta$  is isomorphic to  $\mathbf{R}^{\wedge n}$  and whose generators are represented by the images of the fundamental classes of tori  $T^n$  of dimension  $n$  under the mappings which are defined by  $n$  (commuting) elements. We will construct an  $n$ -complex  $Y_n$  with the fundamental class and a degree one

map  $Y_n \rightarrow T^n$ . Then for each mapping  $T^n \rightarrow B\mathbf{R}^\delta$ , we will construct a mapping  $Y_n \rightarrow BPL_c([0, \infty))^\delta$  such that the following diagram commutes.

$$\begin{array}{ccc} Y_n & \rightarrow & BPL_c([0, \infty))^\delta \\ \downarrow & & \downarrow \\ T^n & \rightarrow & B\mathbf{R}^\delta. \end{array}$$

Theorem (4.1) follows immediately from this commutative diagram.

*Construction of  $Y_n$ .* Let  $L$  be a large positive real number. In the Euclidean  $n$  space, we consider the following polyhedron  $X_n$

$$\begin{aligned} X_n = \{ (x_1, \dots, x_n) \in [0, L]^n ; \quad & x_{i_1} + \dots + x_{i_k} \geq (k-1)k/2 \\ & \text{for } 1 \leq i_1 < \dots < i_k \leq n \}. \end{aligned}$$

The shape of  $X_n$  is the cube with certain neighborhoods of the  $k$ -faces ( $k \leq n-2$ ) in the coordinate planes deleted, those of the  $(k-1)$ -faces being thicker than those of the  $k$ -faces.

The polyhedron  $X_n$  has  $2^n - 1 + n$  faces of dimension  $n-1$ . If  $(x_1, \dots, x_n)$  is a vertex of  $X_n$  then  $(x_1, \dots, x_n)$  is a permutation of  $(0, 1, \dots, k, L, \dots, L)$ . In this case we say  $(x_1, \dots, x_n)$  is a vertex of type  $\{0, 1, \dots, k, L, \dots, L\}$ . There are edges between  $(x_1, \dots, x_n)$  and  $(x'_1, \dots, x'_n)$  of the same type  $\{0, 1, \dots, k, L, \dots, L\}$  if one is obtained from the other by permuting two coordinates. The edges between different types exists only if the types are  $\{0, 1, \dots, k-1, L, \dots, L\}$  and  $\{0, 1, \dots, k, L, \dots, L\}$ , and one vertex is obtained from the other by changing the entries  $k$  and  $L$ .

The polyhedron  $X_n$  has the  $(n-1)$ -face  $\{x_i = L\}$  which is isometric to  $X_{n-1}$ . The  $(n-1)$ -face  $\{x_i = 0\}$  is isometric to  $X_{n-1}$  with  $L$  replaces by  $L-1$  because if  $x_i = 0$  then

$$x_{i_1} + \dots + x_{i_k} \geq (k-1)k/2$$

for  $\{i_1, \dots, i_k\}$  containing  $i$  implies

$$(x_{i_1} - 1) + \dots + (x_{i_k} - 1) \geq (k-1)k/2$$

for  $\{i_1, \dots, i_k\}$  not containing  $i$ . Hence we can define a simplicial identification between the faces  $\{x_i = L\}$  and  $\{x_i = 0\}$ . In general, the face

$$\{x_{i_1} + \dots + x_{i_k} = (k-1)k/2\}$$

is isometric to  $X'_{n-k} \times \Sigma_k$ , where  $X'_{n-k}$  is  $X_{n-k}$  with  $L$  replaced by  $L-k$  and  $\Sigma_k$  is the face  $\{x_1 + \dots + x_k = (k-1)k/2\}$  in  $X_k$ . The reason is

$$x_{i'_1} + \dots + x_{i'_{k'}} \geq (k'-1)k'/2$$

for  $\{i'_1, \dots, i'_{k'}\}$  containing  $\{i_1, \dots, i_k\}$  implies

$$(x_{i'_1} - k) + \dots + (x_{i'_{k'}} - k) \geq (k'-1)k'/2$$

for  $\{i'_1, \dots, i'_{k'}\}$  not containing  $\{i_1, \dots, i_k\}$ . We also fix a simplicial identification between  $X'_{n-k}$  and  $X_{n-k}$ . Now we distinguish the faces by the set  $\{i_1, \dots, i_k\}$  of indices and we see that

$$\begin{aligned} \partial X_n = & \bigcup_{A \subset \{1, \dots, n\}, \# A \geq 2} X_{\{1, \dots, n\} - A} \times \Sigma_A \\ & \cup \bigcup_i X_{\{1, \dots, n\} - \{i\}}^{(L)} \cup \bigcup_i X_{\{1, \dots, n\} - \{i\}}^{(0)}, \end{aligned}$$

where

$$\begin{aligned} X_{\{1, \dots, n\} - A} \times \Sigma_A &= \{x_{i_1} + \dots + x_{i_k} = (k-1)k/2\} \quad \text{if } A = \{i_1, \dots, i_k\}, \\ X_{\{1, \dots, n\} - \{i\}}^{(L)} &= \{x_i = L\} \quad \text{and} \quad X_{\{1, \dots, n\} - \{i\}}^{(0)} = \{x_i = 0\}. \end{aligned}$$

The complex  $Y_n$  is defined inductively as follows.  $Y_1 = X_1 = [0, L]$ .  $Y_2$  is obtained from  $X_2$  (a pentagon) by identifying  $X_{\{i\}}^{(L)}$  and  $X_{\{i\}}^{(0)}$  ( $i = 1, 2$ ) and by taking the double of it. Hence  $Y_2$  is a surface of genus 2. We call the new part in the double  $B\Sigma_{\{1, 2\}}$ .

$$Y_2 = X_2 + B\Sigma_{\{1, 2\}}.$$

$Y_3$  is obtained from  $X_3$  by identifying  $X_{\{i, j\}}^{(L)}$  and  $X_{\{i, j\}}^{(0)}$  ( $i, j = 1, 2, 3$ ), by attaching  $X_{\{k\}} \times B\Sigma_{\{i, j\}}$  ( $\{i, j, k\} = \{1, 2, 3\}$ ) to each  $X_{\{k\}} \times \Sigma_{\{i, j\}}$ , and then by taking the double. The boundary before taking the double is a surface of genus 6. We call the new part in the double  $B\Sigma_{\{1, 2, 3\}}$ .

$$Y_3 = X_3 + \sum_{\{i_1, i_2\} \subset \{1, 2, 3\}} X_{\{1, 2, 3\} - \{i_1, i_2\}} \times B\Sigma_{\{i_1, i_2\}} + B\Sigma_{\{1, 2, 3\}}.$$

In general, we define  $Y_n$  to be the double of

$$X_n + \sum_{A \subset \{1, \dots, n\}, \# A \geq 2} X_{\{1, \dots, n\} - A} \times B\Sigma_A$$

and we call the new part in the double  $B\Sigma_{\{1, \dots, n\}}$ .

$$Y_n = X_n + \sum_{A \subset \{1, \dots, n\}, \# A \geq 2} X_{\{1, \dots, n\} - A} \times B\Sigma_A + B\Sigma_{\{1, \dots, n\}}.$$

The mapping from  $Y_n$  to  $T^n$  is the one which sends the all  $B\Sigma_A$  parts to a point and  $X_n$  to the fundamental domain of  $T^n$ .

*Construction of  $Y_n \rightarrow BPL_c([0, \infty))^{\delta}$ .* Now given a mapping  $T^n \rightarrow B\mathbf{R}^{\delta}$ , we construct a mapping  $Y_n \rightarrow BPL_c([0, \infty))^{\delta}$ . In other words, given a homomorphism  $\mathbf{Z}^n \rightarrow \mathbf{R}$ , we construct a homomorphism  $\pi_1(Y_n) \rightarrow PL_c([0, \infty))$ . This is also done inductively.

For  $n = 1$ , it is only necessary to choose a lift in  $PL_c([0, \infty))$  of an element of  $\mathbf{R}$ .

Now for  $n = 2$ , we choose lifts  $f_1, f_2$  of the generators of  $\mathbf{Z}^2$ . To the edges of  $Y_2$ , we associate elements of  $PL_c([0, \infty))$ . We put  $f_1$  on the edges of  $X_2$  from  $(L, L)$  to  $(0, L)$  and from  $(L, 0)$  to  $(1, 0)$ , and we put  $f_2$  on the edges of  $X_2$  from  $(L, L)$  to  $(L, 0)$  and from  $(0, L)$  to  $(0, 1)$ . Then we put the commutator  $[f_1, f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$  on the edge from  $(0, 1)$  to  $(1, 0)$ . Note that the support of this commutator does not contain 0 hence this commutator is an element of  $PL_c((0, \infty))$ . This commutator is also written as a commutator of elements of  $PL_c((0, \infty))$ . We can do it very easily, not by using the perfectness of the group  $PL_c((0, \infty))$ , but by using a conjugation by an element of  $PL_c(\mathbf{R})$  which sends 0 to  $a (> 0)$  and which is the identity on  $(2a, \infty)$  when the support of  $[f_1, f_2]$  is contained in  $(2a, \infty)$ . We call this conjugation  $c_*$ . (This technique using conjugation is similar to that in [12].)  $c_*$  is an isomorphism from  $PL_c([0, \infty))$  to a subgroup of  $PL_c((0, \infty))$ . Then  $[f_1, f_2] = c_*([f_1, f_2]) = [c_*f_1, c_*f_2]$  and we associate  $c_*f_1, c_*f_2$  to the edges in the new part in the double (in the mirror). Thus we defined the desired mapping  $Y_2 \rightarrow BPL_c([0, \infty))^{\delta}$ .

For general  $n$ , we use the same strategy. First we choose lifts  $f_1, \dots, f_n$  of the generators of  $\mathbf{Z}^n$ . To the edges of  $X_n$ , we associate elements of  $PL_c([0, \infty))$ . We associate  $f_i$  to the edge from a vertex of type  $\{0, 1, \dots, k-1, L, \dots, L\}$  to a vertex of type  $\{0, 1, \dots, k, L, \dots, L\}$  if the  $i$ -th coordinate changes from  $L$  to  $k$ . Then the elements associated to other edges are uniquely determined. In fact, we can associate an element of  $PL_c([0, \infty))$  to each vertices as follows. We associate  $\text{id}$  to the vertex of type  $\{L, \dots, L\}$ , if we already associated an element  $f_v$  to a vertex  $v$  of type  $\{0, 1, \dots, k-1, L, \dots, L\}$  and a vertex  $v'$  is obtained from  $v$  by changing the  $i$ -th coordinate from  $L$  to  $k$  then we associate  $f_i f_v$  to the vertex  $v'$ . Thus the edge from one vertex  $v_1$  to another vertex  $v_2$  is associated with  $f_{v_2} f_{v_1}^{-1}$ . Now if we look at the edges of  $\Sigma_A$  in the  $(n-1)$ -face  $X_{\{1, \dots, n\} - A} \times \Sigma_A$  the associated elements are in  $PL_c((0, \infty))$ . By induction, we can find  $B\Sigma_A$  with edges in  $PL_c((0, \infty))$ . Thus we find the boundary of

$$X_n + \sum_{A \subset \{1, \dots, n\}, \# A \geq 2} X_{\{1, \dots, n\} - A} \times B\Sigma_A$$

is a cycle of  $PL_c((0, \infty))$ . Here the products are considered as in the following remark. Hence in the double  $Y_n$ , we can associate the images under  $c_*$  in the new part of the double. ( $c_*$  is the conjugation by an element of  $PL_c(\mathbf{R})$  which sends 0 to  $a'(> 0)$  and which is the identity on  $(2a', \infty)$  when the support of the above boundary is contained in  $(2a', \infty)$ .) Thus we defined the desired mapping  $Y_n \rightarrow BPL_c([0, \infty))^\delta$ . This proves Theorem (4.1).

*Remark.* For two simplices  $(g_1, \dots, g_m)$  and  $(h_{m+1}, \dots, h_{m+n})$  of the classifying space for a discrete group, we define the product of them as follows.

$$(g_1, \dots, g_m) \times (h_{m+1}, \dots, h_{m+n}) = \sum_{\sigma} \text{sign}(\sigma) (f_{\sigma,1}, \dots, f_{\sigma,m+n}) .$$

where the sum is taken over the shuffles  $\sigma$  (that is, those permutations such that  $\sigma(1) < \dots < \sigma(m)$  and  $\sigma(m+1) < \dots < \sigma(m+n)$ . The entry  $f_{\sigma,j}$  is defined as follows.

$$\begin{aligned} f_{\sigma, \sigma(j)} &= g_j \quad (j = 1, \dots, m) \quad \text{and} \\ f_{\sigma, m+j} &= (g_k \dots g_m) h_{m+j} (g_k \dots g_m)^{-1} \quad (j = 1, \dots, n) , \end{aligned}$$

where  $k$  is the integer such that  $\sigma(k-1) < \sigma(m+j) < \sigma(k)$ . For example,

$$(g_1, g_2) \times (h_3, h_4)$$

$$\begin{aligned} &= (g_1, g_2, h_3, h_4) - (g_1, g_2 h_3 g_2^{-1}, g_2, h_4) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2, h_4) + (g_1, g_2 h_3 g_2^{-1}, g_2 h_4 g_2^{-1}, g_2) \\ &\quad - (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1, g_2 h_4 g_2^{-1}, g_2) \\ &\quad + (g_1 g_2 h_3 (g_1 g_2)^{-1}, g_1 g_2 h_4 (g_1 g_2)^{-1}, g_1, g_2) . \end{aligned}$$

This product is defined so that

$$\begin{aligned} &\partial((g_1, \dots, g_m) \times (h_{m+1}, \dots, h_{m+n})) \\ &= (\partial'(g_1, \dots, g_m)) \times (h_{m+1}, \dots, h_{m+n}) \\ &\quad + (-1)^m (g_1, \dots, g_{m-1}) \times (g_m h_{m+1} g_m^{-1}, \dots, g_m h_{m+n} g_m^{-1}) \\ &\quad + (-1)^m (g_1, \dots, g_m) \times (\partial(h_{m+1}, \dots, h_{m+n})) , \end{aligned}$$

where

$$\begin{aligned} \partial(g_1, \dots, g_m) &= (g_2, \dots, g_m) \\ &+ \sum_{i=1}^{m-1} (-1)^i (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_m) + (-1)^m (g_1, \dots, g_{m-1}) \\ &= \partial'(g_1, \dots, g_m) + (-1)^m (g_1, \dots, g_{m-1}) . \end{aligned}$$

For the above complex we triangulate it and associate the elements for their products.