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Artikel: RATIONALITY OF PIECEWISE LINEAR FOLIATIONS AND HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR HOMEOMORPHISMS

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Kapitel: §4. Construction of cocycles of the group $PL_c(R)$

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This is equivalent to that the differentials induce an isomorphism

$$\sum_{p+q=m,p\geqslant 0} H_{p+1}(B\bar{\Gamma}_1^{PL};\mathbf{Z})\otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z}) \to H_{p+q}(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z}) .$$

To show this we define the cohomology classes of $BPL_c(\mathbf{R})^\delta$ which detect the images of generators of $H_{p+1}(B\bar{\Gamma}_1^{PL};\mathbf{Z})\otimes_{\mathbf{Q}} H_q(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z})$.

§4. Construction of cocycles of the group $PL_c(\mathbf{R})$

Tensor determinants. We define a determinant of an $(n\times n)$ real matrix which takes values in the tensor product over $\mathbf{Q}$ of n copies of $\mathbf{R}$. For $(a_{ij})_{i,j=1,\dots,n}$, we put

$$\det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) = \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{\sigma(1)1}\otimes_{\mathbf{Q}}\cdots\otimes_{\mathbf{Q}} a_{\sigma(n)n} .$$

For example,

$$\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a_{11}\otimes_{\mathbf{Q}} a_{22} - a_{21}\otimes_{\mathbf{Q}} a_{12} .$$

We have the usual multilinearity but we do not have the usual alternativity. For example,

$$\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a & b\\ a & b\end{pmatrix} = 0 \quad \text{but} \quad \det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix} a & a\\ b & b\end{pmatrix} = a\otimes_{\mathbf{Q}} b - b\otimes_{\mathbf{Q}} a = a\wedge_{\mathbf{Q}} b .$$

The latter is not necessarily zero. In general, if we change the rows then this determinant changes sign, however, there are no simple laws for changing columns. It is worth noticing that we have the usual formula of developing with respect to the first or the last column.

$$\begin{aligned}\det{}^{\otimes_{\mathbf{Q}}}(a_{ij}) &= \sum_{i+1}^{n} (-1)^{i+1} a_{i1}\otimes_{\mathbf{Q}} \det{}^{\otimes_{\mathbf{Q}}}(A_{i1})\\ &= \sum_{i+1}^{n} (-1)^{i+n} \det{}^{\otimes_{\mathbf{Q}}}(A_{in})\otimes_{\mathbf{Q}} a_{in} ,\end{aligned}$$

where A_{ij} is the matrix (a_{ij}) with the i-th row and the j-th column deleted.

Cocycles of Lipschitz homeomorphism groups. We review the construction of cocycles of certain Lipschitz homeomorphism groups of the real line or the circle (see [13]). Let $\mathscr{S}$ be the space of functions with compact support which are locally constant outside of finitely many points. (For other Lipschitz homeomorphism groups, $\mathscr{S}$ is replaced by other spaces of functions which contains the logarithm of derivatives of the homeomorphisms.) Let V be a **Q**-vector space. Let

$$A: \overbrace{\mathscr{S} \times \ldots \times \mathscr{S}}^{n} \to V$$

be a multilinear form which is invariant under the parameter change in the following sense. If h is a homeomorphism of **R** with compact support, then

$$A(\varphi_1 \circ h, \ldots, \varphi_n \circ h) = A(\varphi_1, \ldots, \varphi_n) .$$

Then the V valued function

$$C: \overbrace{PL_c(\mathbf{R}) \times \ldots \times PL_c(\mathbf{R})}^{n} \to V$$

defined by

$$C(g_1, g_2, \ldots, g_n) = A(\log g_1' \circ g_2 \circ \ldots \circ g_n, \log g_2' \circ g_3 \circ \ldots \circ g_n, \ldots, \log g_n')$$

is an n-cocycle of $PL_c(\mathbf{R})$. The verification is straightforward.

Cocycles of PL homeomorphism groups. For a $(2s)$-tuple of positive integers $(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)$ such that $k_1^- + k_1^+ + \ldots + k_s^- + k_s^+ = m$, we define a multilinear form

$$A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}: \overbrace{\mathscr{S} \times \ldots \times \mathscr{S}}^{m} \to \mathbf{R}^{\otimes m}$$

whose values are contained in $V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+}$. This is given by

$$A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}(\varphi_1, \ldots, \varphi_m) = \sum_{x_1 < \ldots < x_s} \frac{1}{m!} \det{}^{\otimes_{\mathbf{Q}}}$$

$$(\underbrace{\varphi(x_1 - 0) \ldots \varphi(x_1 - 0)}_{k_1^-} \underbrace{\Delta\varphi(x_1) \ldots \Delta\varphi(x_1)}_{k_1^+}$$

$$\ldots \underbrace{\varphi(x_s - 0) \ldots \varphi(x_s - 0)}_{k_s^-} \underbrace{\Delta\varphi(x_s) \ldots \Delta\varphi(x_s)}_{k_s^+}) ,$$

where φ denotes the vertical vector ${}^t(\varphi_1, \ldots, \varphi_m)$ and $\Delta\varphi(x) = \varphi(x + 0) - \varphi(x - 0)$. Note that, since $\varphi_1, \ldots, \varphi_m$ are elements of $\mathscr{S}$, the sum is in fact a finite sum. It is clear that $A^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ is invariant under the parameter change.

For example, the functional $A^2_{(1,1)}\colon \mathscr{S}\times\mathscr{S}\to \mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}$ is defined as follows.

$$A^2_{(1,1)}(\varphi_1,\varphi_2)=\sum_{x\in\mathbf{R}}\frac{1}{2}\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix}\varphi_1(x-0) & \Delta\varphi_1(x)\\ \varphi_2(x-0) & \Delta\varphi_2(x)\end{pmatrix}$$
$$=\sum_{x\in\mathbf{R}}\frac{1}{2}(\varphi_1(x-0)\otimes_{\mathbf{Q}}\Delta\varphi_1(x)-\varphi_2(x-0)\otimes_{\mathbf{Q}}\Delta\varphi_2(x))\in\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\,.$$

Then $A^2_{(1,1)}$ is bilinear and invariant under the parameter change. This functional $A^2_{(1,1)}$ composed with the evaluation map $\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\to\mathbf{R}$ gives the area of the polygon whose vertices are the image of (φ_1,φ_2) and whose edges join the subsequent vertices with respect to the order of $\mathbf{R}$. The functional $A^2_{(1,1)}$ gives rise to the following 2-cocycle $C^2_{(1,1)}$.

$$C^2_{(1,1)}(g_1,g_2)=\sum_{x\in\mathbf{R}}\frac{1}{2}\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix}\log g_1'\circ g_2(x-0) & \Delta\log g_1'\circ g_2(x)\\ \log g_2'(x-0) & \Delta\log g_2'(x)\end{pmatrix}\in\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\,.$$

This 2-cocycle $C^2_{(1,1)}$ composed with the evaluation map $\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\to\mathbf{R}$ is the discrete Godbillon-Vey invariant ([5], [3], [13], [8]).

The nontriviality of this class is shown easily. Let g_1 and g_2 be piecewise linear homeomorphisms of $\mathbf{R}$ with support in $[-1,0]$ and $[0,1]$, respectively, such that $\log g_1'(0-0)=a$ and $\log g_2'(0+0)=b$. Then $(g_1,g_2)-(g_2,g_1)$ is a 2-cycle and

$$C^2_{(1,1)}((g_1,g_2)-(g_2,g_1))=\frac{1}{2}\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix}a & -a\\ 0 & b\end{pmatrix}-\frac{1}{2}\det{}^{\otimes_{\mathbf{Q}}}\begin{pmatrix}0 & b\\ a & -a\end{pmatrix}$$
$$=a\otimes_{\mathbf{Q}}b\,.$$

Another interesting example is $A^4_{(1,1,1,1)}$ defined by

$$A^4_{(1,1,1,1)}(\varphi_1,\varphi_2,\varphi_3,\varphi_4)=\sum_{x<y}\frac{1}{4!}\det{}^{\otimes_{\mathbf{Q}}}(\varphi(x-0)\,\Delta\varphi(x)\,\varphi(y-0)\,\Delta\varphi(y))$$
$$\in\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{R}$$

where φ denotes the vertical vector ${}^t(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$. This gives rise to the cocycle $C^4_{(1,1,1,1)}$ which measures the noncommutativity of $*$-product. The nontriviality of $C^4_{(1,1,1,1)}$ is easily shown by evaluating on the $*$-product of two examples described above.

Independence of the cohomology classes. The bijectivity of the homomorphism

$$\sum_{p+q=m}H_{p+1}(B\bar{\Gamma}_1^{PL};\mathbf{Z})\otimes_{\mathbf{Q}}H_q(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z})\to H_{p+q}(\Omega B\bar{\Gamma}_1^{PL};\mathbf{Z})$$

is equivalent to the independence of the cohomology classes of these cocycles $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$. To show the independence we use the following theorem.

THEOREM (4.1). *Let* $j_+ : PL_c([0, \infty)) \to \mathbf{R}$ *denote the homomorphism defined by*

$$j_+(f) = \log f'(0) .$$

The homomorphism j_+ *induces a surjection in integer homology.*

Using this theorem, we can show the independence. Let $u_i^- \otimes_{\mathbf{Q}} u_i^+$ be an element of $V^{k_i^-, k_i^+}$ $(u_i^- \in \mathbf{R}^{\wedge k_i^-}, u_i^+ \in \mathbf{R}^{\wedge k_i^+})$. Then we have a k_i^--dimensional cycle σ_i^- of $BPL_c((-\infty, 0])^\delta$ such that the image under $(j_-)_*$ coincides with $u_i^- \in \mathbf{R}^{\wedge k_i^-} \cong H_{k_i^-}(B\mathbf{R}^\delta; \mathbf{Z})$, where $j_- : PL_c((-\infty, 0]) \to \mathbf{R}$ denotes the homomorphism defined by $j_-(f) = \log f'(0)$. We also have a k_i^+-dimensional cycle σ_i^+ of $BPL_c([0, \infty))^\delta$ such that the image under $(j_+)_*$ coincides with $u_i^+ \in \mathbf{R}^{\wedge k_i^+} \cong H_{k_i^+}(B\mathbf{R}^\delta; \mathbf{Z})$. Then $\sigma_i^- \times \sigma_i^+$ is a $(k_i^- + k_i^+)$ -dimensional cycle of $B(PL_c((-\infty, 0]) \times PL_c([0, \infty)))^\delta$ such that the image under $(j_- \times j_+)_*$ coincides with $u_i^- \otimes_{\mathbf{Q}} u_i^+ \in V^{k_i^-, k_i^+}$. Now let $T_1, \ldots, T_s$ be translations of $\mathbf{R}$ such that $T_1(0) < \ldots < T_s(0)$ and the supports of $\sigma_i = T_i(\sigma_i^- \times \sigma_i^+)T_i^{-1}$ are contained in disjoint open intervals, where the support of a cycle of $BPL_c(\mathbf{R})^\delta$ is the union of the supports of the homeomorphisms which appear in the expression of the cycle. Then $\sigma_1 \times \ldots \times \sigma_s$ is an m-cycle and the value of the cocycle $C^m_{(k_1^-, k_1^+, \ldots, k_s^-, k_s^+)}$ on it is $(u_1^- \otimes_{\mathbf{Q}} u_1^+) \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} (u_s^- \otimes_{\mathbf{Q}} u_s^+)$. It is easy to see that the values of the other m-cocycles on this cycle are 0.

The fact that $*$-product coincides with the tensor product follows from Lemma (1.2). Note that the map s in Lemma (1.2) is an isomorphism from the subgroup of $H_*(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$ generated by the $\sigma^- \times \sigma^+$ to $H_{*+1}(B\overline{\Gamma}_1^{PL}; \mathbf{Z})$. Thus Theorem (3.1) is proved.

§5. SURJECTIVITY OF $(j_+)_*$

We prove Theorem (4.1). We consider j_+ as a homomorphism from $PL_c([0, \infty))$ to the group of germs at 0. We use the fact that the n-dimensional homology group of $B\mathbf{R}^\delta$ is isomorphic to $\mathbf{R}^{\wedge n}$ and whose generators are represented by the images of the fundamental classes of tori T^n of dimension n under the mappings which are defined by n (commuting) elements. We will construct an n-complex Y_n with the fundamental class and a degree one