

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 38 (1992)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE CATEGORY OF NILMANIFOLDS  
**Autor:** Oprea, John  
**Kapitel:** §2. Rational homotopy and category  
**DOI:** <https://doi.org/10.5169/seals-59480>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

5. If  $M^{2n}$  is a simply connected compact symplectic manifold, then  $\text{cat}(M) = n = \frac{1}{2} \dim(M)$ . (First, observe that the volume form is not exact since it represents a nontrivial fundamental class of  $M$ . Because  $\omega^n/n! = \text{vol}$  (see [1], p. 165), the nondegenerate closed 2-form  $\omega$  cannot be exact either. Hence,  $\omega^n$  represents a nontrivial cup-product of length  $n$  in  $\mathbf{R}$ -cohomology. By property (4) above,  $\text{cat}(M) \leq (\dim M)/2 = n$ . Hence,

$$n \leq \text{cup}(M) \leq \text{cat}(M) \leq \frac{1}{2} \dim M = n,$$

and the result follows.)

## §2. RATIONAL HOMOTOPY AND CATEGORY

The basic reference for this section is [3]. To each space  $X$ , Sullivan functorially associated a commutative differential graded algebra  $(A(X), d)$  of rational polynomial forms possessing the salient property that integration defines a natural algebra isomorphism between  $H^*(A(X), d)$  and  $H^*(X; \mathbf{Q})$ . Furthermore, the cdga  $A(X)$  was shown to contain all the rational homotopy information about  $X$ ; information which may be gleaned from an associated cdga *minimal model* of  $A(X)$ .

A cdga  $(\Lambda, d)$  is *minimal* if (1)  $\Lambda = \Lambda X$ , where  $X = \bigoplus_{i>0} X^i$  is a graded  $\mathbf{Q}$ -vector space and  $\Lambda X$  denotes that  $\Lambda$  is freely generated by  $X$ ; that is,  $\Lambda X = \text{Symmetric algebra}(X^{\text{even}}) \otimes \text{Exterior algebra}(X^{\text{odd}})$ . (2) There is a basis for  $X$ ,  $\{x_\alpha\}_{\alpha \in I}$ , so that if  $I$  is well ordered by  $<$ , then  $dx_\beta \in \Lambda_{\alpha < \beta}^+(x_\alpha) \cdot \Lambda_{\alpha < \beta}^+(x_\alpha)$ . That is,  $\Lambda$  is constructed by stages and the differentials of  $\beta^{\text{th}}$  stage generators are decomposable in the generators of previous stages.

A *minimal model* for a space  $M$  is a minimal cdga  $\Lambda(M)$  and a cdga map  $\Lambda(M) \rightarrow A(M)$  inducing an isomorphism in cohomology. The fundamental theorem of rational homotopy theory is then (see [4] for example).

**THEOREM.** *Each space  $M$  has a minimal model  $\Lambda(M)$  and, furthermore, for nilpotent spaces the stage by stage construction precisely mirrors the rational Postnikov tower with the differential corresponding to the  $k$ -invariant.*

Recall that a space  $M$  is *nilpotent* if its fundamental group  $\pi_1(M)$  is a nilpotent group and the natural action of  $\pi_1(M)$  on  $\pi_n(M)$  (see [10]) is a nilpotent action (see [12]). In particular, any simply connected space or any  $K(\pi, 1)$  with  $\pi$  nilpotent is a nilpotent space. The theorem then says that, for

a nilpotent space, the minimal model is a perfect reflection of the rational homotopy type of the space (eg for  $i > 1$ ,  $X^i \cong \text{Hom}(\pi_i(M), \mathbf{Q})$ , where  $\pi_i(M)$  is the  $i^{\text{th}}$  homotopy group of  $M$ ). The minimal model  $\Lambda(M)$  is therefore an algebraic version of the  $\mathbf{Q}$ -localization of  $M$ . Indeed, a notion of cdga homotopy may be described so that there is a categorical equivalence between the homotopy categories of rational nilpotent spaces and minimal cdga's.

- Examples.* (1)  $\Lambda(S^{2n+1}) = \Lambda(x_{2n+1}), dx = 0$ .  
 (2)  $\Lambda(S^{2n}) = \Lambda(x_{2n}, y_{4n-1}), dy = x^2$ .  
 (3)  $\Lambda(\mathbf{C}P(n)) = \Lambda(x_2, y_{2n+1}), dy = x^{n+1}$ .  
 (4)  $\Lambda(T^n) = \Lambda(x_1^1, x_1^2, \dots, x_1^n), d = 0$ .

In the next section we will describe the minimal model of a nilmanifold in terms of the structure of its defining nilpotent group.

In order to understand category in the framework of minimal models, assume for the moment that  $\text{cat}(M) = m$ . The Whitehead diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\Delta} & M^{m+1} \\
 (*) & \Delta' \searrow & \uparrow j \\
 & & T^{m+1}(M)
 \end{array}$$

translates (via Sullivan's categorical equivalence) into a homotopy commutative diagram of minimal cdga's,

$$\begin{array}{ccc}
 \Lambda X & \leftarrow & (\Lambda X)^{\otimes m+1} \\
 (**) & \rho \swarrow & \downarrow \xi \\
 & & \Lambda Y
 \end{array}$$

where  $\Lambda(M) = \Lambda X$ ,  $\Lambda(M^{m+1}) = (\Lambda X)^{\otimes m+1}$  (since the model of a product is the tensor product of the models),  $\Delta$  is modelled by the  $(m + 1)$ -fold multiplication  $\mu$  and  $\Lambda Y = \Lambda(T^{m+1}(M))$ .

Now, however, we may make the following

*Definition.* The *rational category* of  $M$  (or  $\Lambda(M) = \Lambda X$ ),  $\text{cat}_0(M)$ , is the least  $m$  so that  $(**)$  exists; that is, there exists  $\rho$  with  $\rho\xi \simeq \mu$ .

Observe that: (1)  $\text{cat}_0(M) \leq \text{cat}(M)$  since any diagram  $(*)$  induces a diagram  $(**)$ . (2) If  $M$  is simply connected, then  $\text{cat}_0(M) = \text{cat}(M_0)$ , where  $M_0$  is the  $\mathbf{Q}$ -localization of  $M$ . This follows since  $(*)$  itself localizes.

The definition of  $\text{cat}_0(M)$  would be of little use if this were its only description. The passage from (\*) to (\*\*) simply transfers the difficult problem of obtaining  $\Delta'$  to an (almost) equally difficult problem of obtaining  $\rho$ . However, by understanding the nature of  $\Lambda Y = \Lambda(T^{m+1}(M))$ , a more accessible criterion for  $\text{cat}_0(M)$  may be developed. We first describe  $\Lambda Y$ .

PROPOSITION (2.2 of [3]). *A minimal model for the fat wedge is given by a minimal model  $\phi: \Lambda Y \rightarrow \Omega$  for the quotient cdga*

$$\Omega = (\Lambda X)^{\otimes m+1} / (\Lambda^+ X)^{\otimes m+1}$$

where  $\Lambda^+ X$  consists of all elements of positive degree. Moreover, if  $\pi: (\Lambda X)^{\otimes m+1} \rightarrow \Omega$  is the projection, then any  $\eta: (\Lambda X)^{\otimes m+1} \rightarrow \Lambda Y$  with  $\phi\eta \simeq \pi$  is homotopic to the induced map  $\xi$ .

(The existence of  $\eta$  is a consequence of the minimality of  $(\Lambda X)^{\otimes m+1}$ , the fact that  $\phi$  induces an isomorphism of cohomology and cdga obstruction theory. See [4] or [6].)

In some sense, the form of  $\Omega$  is exactly what one would expect viewing the fat wedge as a spatial bound on the “form product” length (as opposed to cuplength). The proof of the proposition relies on various technical results involving  $A(T^{m+1}(M))$ .

Now let  $\Lambda^{>m} X$  denote the differential ideal of  $\Lambda X$  having additive basis the monomials  $x_{i_1} \cdots x_{i_k}$  with  $k > m$ . Consider the projection  $p: \Lambda X \rightarrow \Lambda X / \Lambda^{>m} X$  and a minimal model  $\theta: \Lambda Z \rightarrow \Lambda X / \Lambda^{>m} X$ . As before (for  $\Lambda Y$ ), minimal model theory provides a lift of  $p, \tilde{p}: \Lambda X \rightarrow \Lambda Z$ , with  $\theta\tilde{p} \simeq p$ .

Say that  $\Lambda X$  is a *retract* of  $\Lambda X / \Lambda^{>m} X$  if there exists a cdga map  $r: \Lambda Z \rightarrow \Lambda X$  with  $r\tilde{p} \simeq 1_{\Lambda X}$ .

We are now in a position to give the rational homotopy criterion for category. We give a proof in one direction and refer to [3] for the other. (Also, we make use of the fact that a cohomology isomorphism  $\theta: A \rightarrow B$  induces bijections of cdga homotopy sets  $\theta_*: [\Lambda, A] \xrightarrow{\cong} [\Lambda, B]$  for any minimal  $\Lambda$ .) With the notation above, we have the

THEOREM.  $\text{cat}_0(M) \leq m$  if and only if  $\Lambda X = \Lambda(M)$  is a retract of  $\Lambda X / \Lambda^{>m} X$ .

*Proof.* We only prove the “if” part. Let  $r$  denote the retraction,  $\Lambda Z \rightarrow \Lambda X$ , with  $r\tilde{p} \simeq 1_{\Lambda X}$ . We have the following homotopy commutative diagram (where  $\bar{\mu}$  is the map induced by  $\mu$  and  $\tilde{\mu}$  is a lift to models),

$$\begin{array}{ccc}
 \Lambda X & \xleftarrow{\mu} & (\Lambda X)^{\otimes m+1} \\
 \downarrow p & & \downarrow \pi \\
 \frac{\Lambda X}{\Lambda^{>m} X} & \xleftarrow{\bar{\mu}} & \frac{(\Lambda X)^{\otimes m+1}}{(\Lambda + X)^{\otimes m+1}} \\
 \simeq \uparrow \theta & & \simeq \uparrow \phi \\
 \Lambda Z & \xleftarrow{\tilde{\mu}} & \Lambda Y
 \end{array}
 \begin{array}{l}
 \tilde{p} \\
 \xi
 \end{array}$$

In order to prove  $\text{cat}_0(M) \leq m$ , we must find  $\rho: \Lambda Y \rightarrow \Lambda X$  with  $\rho\xi \simeq \mu$ . We can use the given retraction to do exactly this. Let  $\rho = r\tilde{\mu}$ .

First, observe  $\theta\tilde{p}\mu \simeq p\mu = \bar{\mu}\pi \simeq \bar{\mu}\phi\xi \simeq \theta\tilde{\mu}\xi$ . Because  $\theta$  is a cohomology isomorphism,  $\tilde{p}\mu \simeq \tilde{\mu}\xi$ .

Now,  $\rho\xi = r\tilde{\mu}\xi \simeq r\tilde{p}\mu \simeq 1_{\Lambda X}\mu = \mu$  and we are done.  $\square$

Of course,  $\text{cat}_0(M)$  is, in general, too hard to compute. However, the criterion we have described opens up the possibility of defining weaker invariants which *are* computable. In a sense, the point of this paper is to give an exposition of these weaker invariants in the context of a specific problem of interest to ‘‘geometers’’.

Define  $e_0(M)$  to be the least integer  $s$  so that  $p: \Lambda X \rightarrow \Lambda X/\Lambda^{>s} X$  induces an injection in cohomology. (This is, in fact, equivalent to requiring  $r: \Lambda Z \rightarrow \Lambda X$  to be only a *linear* retraction. The invariant  $e_0(M)$  was first defined by Toomer [9] in terms of the Milnor-Moore spectral sequence.)

Note that if  $r: \Lambda Z \rightarrow \Lambda X$  is a retraction, then  $\tilde{p}^*$  is injective and (since  $\theta^*$  is an isomorphism) therefore so is  $p^*$ . Hence, we clearly have

$$e_0(M) \leq \text{cat}_0(M) .$$

Moreover, when  $M$  is a nilpotent space (so that the full power of the minimal model may be utilized) *and* a manifold (so that Poincaré duality may be exploited), we can identify  $e_0(M)$  in the following manner:

**PROPOSITION.** *If  $M^n$  is a nilpotent manifold with fundamental class  $\tau \in H^n(M; \mathbf{Q})$ , then  $e_0(M)$  is the largest  $k$  such that  $\tau$  is represented by a cocycle in  $\Lambda^{\geq k} X$ .*

*Proof.* Let  $e_0(M) = s$  and let  $k$  be defined by the stated property. If  $\tau$  is represented by a cocycle in  $\Lambda^{>s} X$ , then (for  $p: \Lambda X \rightarrow \Lambda X/\Lambda^{>s} X$ )  $p^*(\tau) = 0$  and  $p^*$  is therefore not injective. Hence,  $k \leq s$ .

In order to show the reverse inequality  $s \leq k$ , we must show that, for  $p: \Lambda X \rightarrow \Lambda X / \Lambda^{>k} X$ ,  $p^*$  is injective. Plainly, by Poincaré duality,  $p^*$  is injective if and only if  $p^*(\tau) \neq 0$ . Hence, we prove this.

Suppose  $p^*(\tau) = 0$ . Let  $\tau$  denote the representing cocycle in  $\Lambda^{\geq k} X$  of the fundamental class  $\tau$ . Let  $p(\tau) = \bar{\tau} \in \Lambda X / \Lambda^{>k} X$  and consider  $\bar{\tau}$  as an element in  $\Lambda^{\leq k} X$ . Now,  $p^*(\tau) = 0$ , so there exists  $\bar{\alpha} \in \Lambda X / \Lambda^{>k} X$  with  $d\bar{\alpha} = \bar{\tau}$ . Consider  $\bar{\alpha} \in \Lambda^{\leq k} X$  as well and note that  $p(d\bar{\alpha}) = d\bar{\alpha} = \bar{\tau}$ . Therefore, in  $\Lambda X$

$$d\bar{\alpha} = \bar{\tau} + \Phi, \quad \text{where } \Phi \in \Lambda^{>k} X.$$

Similarly, of course,  $\tau = \bar{\tau} + \Omega$  for  $\Omega \in \Lambda^{>k} X$  and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with  $\Omega - \Phi \in \Lambda^{>k} X$ . But this means  $\tau$  is cohomologous to  $\Omega - \Phi \in \Lambda^{>k} X$ , contradicting the definition of  $k$ .  $\square$

### §3. NILMANIFOLDS

A *nilmanifold*  $M$  is the quotient of a nilpotent Lie group  $N$  by a discrete cocompact subgroup  $\pi$ . The description below follows [7].

It is well known that  $N$  is diffeomorphic to some  $\mathbf{R}^n$  and, therefore,  $M$  is a  $K(\pi, 1)$ . Furthermore, this entails the fact that  $\pi$  is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of  $\pi$ ,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each  $\pi_i / \pi_{i+1} \cong \mathbf{Z}$  whose length is invariant and is called the *rank* of  $\pi$ . So, for  $\pi$  above,  $\text{rank}(\pi) = n$ .

This description implies that any  $u \in \pi$  has a decomposition  $u = u_1^{x_1} \cdots u_n^{x_n}$ , where  $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i / \pi_{i+1}$ . The set  $\{u_1, \cdots, u_n\}$  is called a Malcev basis for  $\pi$ . Using this basis the multiplication in  $\pi$  takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x, y)} \cdots u_n^{\rho_n(x, y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1}).$$