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**Artikel:** RATIONALITY OF PIECEWISE LINEAR FOLIATIONS AND  
HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR  
HOMEOMORPHISMS

**Autor:** Tsuboi, Takashi  
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$$Y \times Z \rightarrow \Omega X \times \Omega X \xrightarrow{*} \Omega X .$$

Then this represents  $u * v \in H_{p+q}(\Omega X; \mathbf{Z})$ . On the other hand, the composition

$$Y \times Z \rightarrow \Omega X \times \Omega X \rightarrow PX \times \Omega X \rightarrow X \times \Omega X$$

bounds  $SY \times Z \rightarrow PX \times \Omega X \rightarrow X \times \Omega X$ , which represents  $s(u) \otimes v$ . Hence  $s(u) \otimes v$  and  $u * v$  are related under  $\partial^{p+1}$ .

§2. DISCONTINUOUS INVARIANTS

First we review the definition by Morita ([10]) of discontinuous invariants arising from the Godbillon-Vey invariant for codimension one foliations.

Let  $\mathcal{F}$  be a codimension one foliation of a closed oriented  $3k$ -dimensional manifold  $M$ . Then the Godbillon-Vey class  $gv(\mathcal{F}) \in H^3(M; \mathbf{R})$  is defined ([6]). Let  $\{x_1, \dots, x_n\}$  be a basis of  $H^3(M; \mathbf{Q})$ . Then  $gv(\mathcal{F})$  is written as

$$gv(\mathcal{F}) = a_1 x_1 + \dots + a_n x_n ,$$

where  $a_1, \dots, a_n \in \mathbf{R}$ . The discontinuous invariant  $GV_k$  is defined by

$$GV_k(\mathcal{F}) = \sum_{i_1 < \dots < i_k} (x_{i_1} \cup \dots \cup x_{i_k}) [M] a_{i_1} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} a_{i_k} \in \mathbf{R}^{\wedge k} = \overbrace{\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R}}^k ,$$

where  $[M] \in H_{3k}(M; \mathbf{Z})$  is the fundamental class. Morita showed that  $GV_k$  is natural,  $GV_k$  depends only on the foliated cobordism class of  $\mathcal{F}$ , and hence there is a universal map  $GV_k: H_{3k}(B\Gamma_1; \mathbf{Z}) \rightarrow \mathbf{R}^{\wedge k}$  ([10]).

The same argument applies to transversely piecewise linear foliations and the discrete Godbillon-Vey class defined in [5] and [3]. Then the following theorem is obtained from the description by Greenberg ([7]) of the classifying space for them and Lemma (1.1).

**THEOREM (2.1).** *Let  $\mathcal{F}$  be a codimension one transversely orientable transversely piecewise linear foliation of a closed oriented  $3k$ -dimensional manifold  $M(k \geq 2)$ . Then  $GV_k(\mathcal{F}) = 0$ .*

*Proof.* The weak homotopy type of the classifying space  $B\bar{\Gamma}_1^{PL}$  for codimension one transversely oriented transversely piecewise linear foliations is known by Greenberg ([7]). This classifying space  $B\bar{\Gamma}_1^{PL}$  has the weak homotopy type of the join  $B\mathbf{R}^\delta * B\mathbf{R}^\delta$  of two copies of  $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$ . Let

$gv$  denote the discrete Godbillon-Vey class defined as a 3-dimensional cohomology class of this classifying space ([5], [3]).

$$gv \in H^3(B\bar{\Gamma}_1^{PL}; \mathbf{R}) .$$

By Lemma (1.1), the higher discontinuous invariants  $GV_k$  are trivial in this classifying space  $B\bar{\Gamma}_1^{PL}$ . Hence by the naturality of  $GV_k$ ,  $GV_k(\mathcal{F}) = 0$ .

**COROLLARY (2.2).** *Let  $\mathcal{F}$  be a codimension one transversely piecewise linear foliation of  $S^3 \times S^3$ .  $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3, \mathbf{R})$  satisfies  $a/b \in \mathbf{Q} \cup \{\infty\}$ .*

*Proof.*  $0 = GV_2(\mathcal{F}) = a \wedge_{\mathbf{Q}} b$ . Hence  $a/b \in \mathbf{Q} \cup \{\infty\}$ .

*Remark.* Morita translated the question of rationality into that of graded commutativity of  $*$ -product defined on the homology of the group of diffeomorphisms of  $\mathbf{R}$  with compact support ([10]). In the later sections, we calculate the homology of the group  $PL_c(\mathbf{R})$  of piecewise linear homeomorphisms of  $\mathbf{R}$  with compact support as well as the  $*$ -product structure. We see that the  $*$ -product is certainly not graded commutative, which insures the rationality. The argument on the rationality of transversely piecewise linear foliations uses the fact that the Godbillon-Vey invariant localizes on transversely discrete sets and this argument cannot be generalized for smooth foliations for the moment. See how the class  $C_{(1,1,1,1)}^4$  exists in §3. We also see that the Whitehead product of elements of  $\pi_n(B\bar{\Gamma}_1^{PL})$  which are not zero in homology is usually nontrivial and has infinite order.

*Remark.* The Hurewicz map

$$\pi_n(B\bar{\Gamma}_1^{PL}) \rightarrow H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

is surjective. To see this, note first that by Greenberg ([7]),

$$H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \cong \sum_{i=1}^{n-1} \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i} .$$

An element  $(a_1 \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} a_i) \otimes_{\mathbf{Q}} (b_{i+1} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} b_{n-1}) \in \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i}$  is represented by the following foliation of  $T^i * T^{n-1-i}$ . Consider the foliated  $\mathbf{R}$ -product with noncompact support over  $T^{n-1}$  such that the holonomy  $h: \pi_1(T^{n-1}) \rightarrow PL(\mathbf{R})$  is given by

$$\begin{aligned} h(e_j)(x) &= e^{a_j}x \text{ for } x < 0 \text{ and } h(e_j)(x) = x \text{ for } x > 0 \text{ if } j = 1, \dots, i \\ h(e_j)(x) &= x \text{ for } x < 0 \text{ and } h(e_j)(x) = e^{b_j}x \text{ for } x > 0 \text{ if } j = i + 1, \dots, n - 1 . \end{aligned}$$

This foliation restricted to  $T^{n-1} \times [-1, 1]$  induces a foliation of  $T^i * T^{n-1-i}$  which is

$$T^{n-1} \times [-1, 1] / (T^i \times T^{n-1-i} \times \{-1\} \sim T^i \times \{-1\}, \\ T^i \times T^{n-1-i} \times \{1\} \sim T^{n-1-i} \times \{1\}).$$

Note that there is a degree one map from the suspension of  $T^{n-1}$  to  $T^i * T^{n-1-i}$ . Since we can embed  $T^{n-1} \times [-1, 1]$  in  $S^n$ , we have a degree one map from  $S^n$  to the suspension of  $T^{n-1}$ , hence to  $T^i * T^{n-1-i}$ . Thus Hurewicz map is surjective.

### §3. HOMOLOGY OF THE GROUP OF PIECEWISE LINEAR HOMEOMORPHISMS

Let  $PL_c(\mathbf{R})$  denote the group of piecewise linear homeomorphisms of  $\mathbf{R}$  with compact support. Let  $\mu: PL_c(\mathbf{R}) \times PL_c(\mathbf{R}) \rightarrow PL_c(\mathbf{R})$  be the composition of two isomorphisms  $PL_c(\mathbf{R}) \cong PL_c((-\infty, 0))$  and  $PL_c(\mathbf{R}) \cong PL_c((0, \infty))$ , and the inclusion

$$PL_c((-\infty, 0)) \times PL_c((0, \infty)) \rightarrow PL_c(\mathbf{R}).$$

Then  $\mu$  induces a product  $*$  on the homology of  $BPL_c(\mathbf{R})^\delta$  ([10]).

The homology of the group  $PL_c(\mathbf{R})$  of piecewise linear homeomorphisms of  $\mathbf{R}$  with compact support is described as follows. For positive integers  $i$  and  $j$ , put

$$V^{i,j} = \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge j} \\ = \underbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}_i \otimes_{\mathbf{Q}} \underbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}_j.$$

THEOREM (3.1).

$$H_m(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \sum V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \dots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+},$$

where the sum is taken over even number of positive integers

$$(k_1^-, k_1^+, \dots, k_s^-, k_s^+)$$

such that  $k_1^- + k_1^+ + \dots + k_s^- + k_s^+ = m$ . Moreover, the  $*$ -product

$$*: H_i(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \times H_j(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \rightarrow H_{i+j}(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$$

coincides with the tensor product.