

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 38 (1992)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI
AND PROGRESSION-FREE SEQUENCES OF INTEGERS
Autor: Allouche, Jean-Paul / Bacher, Roland
Kapitel: 3. Iteration of continuous functions and Toeplitz transforms
DOI: <https://doi.org/10.5169/seals-59494>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. ITERATION OF CONTINUOUS FUNCTIONS AND TOEPLITZ TRANSFORMS

When iterating a unimodal (i.e. increasing then decreasing) and continuous map of the interval $[0, 1]$, say F_μ , depending on the parameter μ , one knows that, assuming certain properties of the map $\mu \rightarrow F_\mu$, a Feigenbaum doubling cascade phenomenon occurs (see [7] for instance): when the parameter increases, the function has first an attractive fixed point for $\mu_0 \leq \mu < \mu_1$, then an attractive cycle of length 2 for $\mu_1 \leq \mu < \mu_2$, then an attractive cycle of length 4, ... There is a "first" value of the parameter μ_∞ for which a "chaotic" behaviour appears, and this value is the limit of the sequence $(\mu_n)_n$. This sequence grows roughly like a constant term plus a geometric progression C^n , where the constant C is universal, provided that the functions F_μ are smooth enough. This constant is called the Feigenbaum constant.

The orbit of the point 1 under F_{μ_∞} can be coded by a universal binary sequence A (even in cases where the Feigenbaum constant does not appear). This sequence $A = (A(n))$ is defined as 0 if $F_{\mu_\infty}^{(n)}(1)$ is smaller than the point where F_{μ_∞} takes its maximum, and 1 if $F_{\mu_\infty}^{(n)}(1)$ is larger than this value, and does not depend on the family of functions (F_μ) . Moreover it has been noticed in [3] that the sequence A is related to the Prouhet-Thue-Morse sequence C (see [6] and its bibliography) by

$$C(n+1) = \sum_{0 \leq j \leq n} A(j) \text{ modulo } 2.$$

(Let us recall that C is the fixed point beginning by 0 of the 2-substitution $0 \rightarrow 01, 1 \rightarrow 10$.)

Actually as noticed in [24], A is the fixed point of the 2-substitution

$$1 \rightarrow 10, \quad 0 \rightarrow 11.$$

PROPOSITION. *Let f be defined by $f(0) = 1, f(1) = 0, f(\omega) = \omega$, then the sequence A is the Toeplitz transform of $((1\omega)^\infty, f)$.*

Proof. As the fixed point of the 2-substitution $1 \rightarrow 10, 0 \rightarrow 11$, the sequence A can be recursively defined by

$$A(2n) = 1, \quad A(2n+1) = 1 - A(n).$$

Remark. The relation between C and A can also be written

$$A(n) = C(n) + C(n+1) \text{ modulo } 2.$$

If, instead of C one takes a “generalized” Morse sequence C' , and if one defines

$$A'(n) = C'(n) + C'(n+1) \text{ modulo } 2,$$

then A' is also a Toeplitz sequence, as proved in [16].

4. TOWERS OF HANOI AND TOEPLITZ SEQUENCES

The tower of Hanoi puzzle consists of three vertical pegs and of N circular disks of different diameters stacked in decreasing order on the first peg. At each step one may transfer the topmost disk from a peg to a different peg according to the rule: no disk is allowed to be on a smaller one. The game ends when all the disks are stacked on the second or third peg.

The sequence of moves for the classical (minimal) Hanoi tower algorithm can be generated in a very easy way as it is 2-automatic (see [4] and section 6), which essentially means that the k^{th} move can be predicted by a machine with bounded memory. More precisely number the pegs as I, II, III and define a (respectively b, c) to be the move which takes the topmost disk from peg I (respectively II, III) and puts it on peg II (respectively III, I). Let $\bar{a}, \bar{b}, \bar{c}$ be the respective opposite moves. Then the sequence of moves for N disks is the prefix of length $2^N - 1$ of an infinite sequence U which is 2-automatic. Moreover the following proposition is proved in [4]:

PROPOSITION. *The infinite sequence of moves U is equal to the Toeplitz transform of $((a\bar{c}b\omega c\bar{b}a\omega b\bar{a}c\omega)^\infty, id)$.*

Note that, keeping the notations of [4], the sequence U is indexed by $1, 2, \dots$ and not by $0, 1, 2, \dots$ as the sequences above.

5. PROGRESSION-FREE SEQUENCES AND TOEPLITZ SEQUENCES

The question of finding a sequence of integers without arithmetic progressions of given length has been intensively studied (see its history in [14] and the included bibliography). In particular what is the “minimal” increasing sequence having this property?

One knows that, if k is a prime number, the minimal sequence of integers without any arithmetic progression of k terms is exactly the increasing sequence of the integers without the digit $k - 1$ in their base k expansion (cited