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Artikel: TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

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Kapitel: 2. Paperfolding sequences and Toeplitz transforms

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Example. Let B be the sequence $(0\omega 1\omega)^\infty$, and let f be defined on $\{0, 1, \omega\}$ by $f(0) = 1$, $f(1) = 0$, $f(\omega) = \omega$, then one has:

$$B_0 = (0\omega 1\omega 0\omega 1\omega \cdots)\ (= B)\ ,$$
$$B_1 = (1\omega 0\omega 1\omega 0\omega \cdots)\ ,$$
$$B_2 = B_0\ ,$$
$$\ldots$$
$$A_0 = (0\omega 1\omega 0\omega 1\omega \cdots)\ ,$$
$$A_1 = (011\omega 001\omega \cdots)\ ,$$
$$A_2 = (0110001\omega \cdots)\ ,$$
$$\ldots\ .$$

Note that if Γ is finite, and f one-to-one, such a sequence $Tt(B, f)$ can also be obtained by replacing B by a sequence of greater period and f by *id*.

We now give four examples of Toeplitz transforms in (apparently) unrelated domains.

2. Paperfolding sequences and Toeplitz transforms

In [23] and [22] Prodinger and Urbanek study the Toeplitz transform of $((0\omega 1\omega)^\infty, id)$ and of $((0\omega 1\omega 1\omega 0\omega)^\infty, id)$. They prove that these sequences do not have arbitrarily long squares (a sequence A contains a square of length $2k$ if there exists an index j such that $A(j+n) = A(j+n+k)$ for every n between 0 and $k-1$). Dekking already noticed in [10] that the first sequence is nothing but the regular paperfolding sequence (see [9], [18], [20], [17]), which is obtained by repeatedly folding a piece of paper, and we obtained in [1] the same result as Prodinger and Urbanek for the general paperfolding sequences. Let us give here two simple examples:

PROPOSITION. *Let B be the sequence $B = (0\omega 1\omega)^\infty$ and let f be defined by $f(0) = 1$, $f(1) = 0$ and $f(\omega) = \omega$. Then*

the sequence $Tt(B, id)$ is the regular paperfolding sequence,

the sequence $Tt(B, f)$ is the alternate paperfolding sequence.

Proof. It follows from instance from [18] (after replacing 1's by 0's and -1's by 1's) that the regular paperfolding sequence R and the alternate paperfolding sequence A are given by

$$R(2^k(2m+1)-1) = \frac{(1-(-1)^m)}{2} \quad \forall k, m \geqslant 0 .$$

$$A(2^k(2m+1)-1) = \frac{(1-(-1)^{k+m})}{2} \quad \forall k, m \geqslant 0 .$$

Let U and V be the sequences defined by

$$U = Tt((0\omega\, 1\omega)^\infty, id) ,$$

$$V = Tt((0\omega\, 1\omega)^\infty, f) .$$

A straightforward computation gives

$$U(2n) = \frac{(1-(-1)^n)}{2} \quad \forall n \geqslant 0 ,$$

$$U(2n+1) = U(n) \quad \forall n \geqslant 0 .$$

Hence

$$U(2^k(2m+1)-1) = U(2(2^{k-1}(2m+1)-1)+1)$$

$$= U(2^{k-1}(2m+1)-1) = \cdots = U(2m) = \frac{(1-(-1)^m)}{2} .$$

This proves that $U = R$.

In the same way one has

$$V(2n) = \frac{(1-(-1)^n)}{2} ,$$

$$V(2n+1) = 1 - V(n) .$$

Hence

$$V(2^k(2m+1)-1) = V(2(2^{k-1}(2m+1)-1)+1) = 1 - V(2^{k-1}(2m+1)-1)$$

$$= \cdots = \left\{ \begin{array}{ll} V(2m) & \text{if } k \text{ is even,} \\ 1 - V(2m) & \text{if } k \text{ is odd,} \end{array} \right\} = \frac{(1-(-1)^{k+m})}{2} ,$$

and finally $V = A$.