Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	38 (1992)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	THE CATEGORY OF NILMANIFOLDS
Autor:	Oprea, John
Kapitel:	§1. Category
DOI:	https://doi.org/10.5169/seals-59480

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

## Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

### J. OPREA

THEOREM 1. If M is a (compact) nilmanifold, then  $cat(M) = dim(M) = rank(\pi_1 M)$ .

Hence, the best possible result which Lusternik-Schnirelmann theory can provide for nilmanifolds is the immediate.

COROLLARY. The number of critical points of a smooth function on a (compact) nilmanifold M is bounded below by  $\operatorname{rank}(\pi_1 M) + 1$ .

In fact, Theorem 1 was announced for all  $K(\pi, 1)$ 's by Eilenberg and Ganea [11]. Unfortunately, details of the proofs of their three fundamental propositions never appeared, thus contributing, I believe, to the ignorance of the result among the dynamicists and topologists of today. Indeed, this paper was originally written in response to Chris McCord's question and without knowledge of the Eilenberg-Ganea result. Furthermore, in looking at the Eilenberg-Ganea propositions, it is difficult to see the relationship between the structures of  $\pi$  and  $K(\pi, 1)$  and the consequent determination of category as rank( $\pi$ ). I hope that the approach of this paper will remedy this defect, at least in the case of nilmanifolds. The beautiful structure theory of nilmanifolds (i.e. finitely generated torsionfree nilpotent groups) is ideally suited for an approach in terms of minimal models. In fact, in some sense, this paper is simply an exposition of just how well rational homotopy theory and nilmanifold theory fit together (in the representative situation of determining category).

Theorem 1 will be given a simple ("up to" the machinery of rational homotopy theory) proof in §4. Since this paper is written for workers in dynamical systems, I have tried to make it somewhat self-contained. Therefore, §1 and §2 are devoted to recollections on category and its rational homotopy description respectively. §3 recollects structural knowledge of nilmanifolds and §5 presents an analogue of Theorem 1 for iterated principal bundles. (The basic reference for the rational homotopy version of L.S. category is [3]; I have attempted to cull the essential ingredients for the proof of Theorem 1, but the reader will find other interesting applications in that work. Also see [2].)

# §1. CATEGORY

The category of a space M, cat(M), is the least integer m so that M is covered by m + 1 open subsets each of which is contractible within M.

An equivalent definition (at least for the spaces we consider here) was given by G. Whitehead (see [10]): Let  $M^{m+1}$  denote the (m + 1)-fold product and let  $T^{m+1}(M)$  denote the subspace consisting of all (m + 1)-tuples  $(x_1, \dots, x_{m+1})$  with at least one  $x_i$  equal to a specified basepoint in M.  $(T^{m+1}(M)$  is usually called the "fat wedge".) In particular,  $T^2(M) = M \vee M$ ; two copies of M attached at the specified basepoint. Now let  $\Delta: M \to M^{m+1}$ denote the (m + 1)-fold diagonal  $\Delta(x) = (x, x, \dots, x)$  and  $j: T^{m+1}(M) \to M^{m+1}$  the natural inclusion. Whitehead's definition is then: cat(M) is the least integer m so that, up to homotopy,  $\Delta$  factors through the fat wedge; that is, there exists  $\Delta': M \to T^{m+1}(M)$  with  $j \Delta' \simeq \Delta$ .

The cuplength of M, cup(M), is the largest integer k so that there exist  $x_i \in H^{n_i}(M; R)$ ,  $i = 1, \dots, k$  and a nontrivial cup-product

$$0\neq x_1x_2\cdots x_k.$$

The following result is well-known and is the basis of many calculations of category:

PROPOSITION.  $\operatorname{cup}(M) \leq \operatorname{cat}(M)$ .

For a proof, see [10] for example. Other important properties of category are:

(1) Category is an invariant of homotopy type.

(2) If  $C_f = Y \cup_f CX$  is a mapping cone, then  $\operatorname{cat}(C_f) \leq \operatorname{cat}(Y) + 1$ .

(3) If X is a CW-complex, then (by induction on skeleta and (2))  $cat(X) \le \dim X$ .

(4) In fact, (3) may be generalized: If X is (r-1)-connected, then  $cat(X) \leq (\dim X)/r$ .

The proofs of these properties are straightforward; see [10] for example. In particular, we shall use (3) in our determination of the category of nilmanifolds.

## Examples

1. cat(X) = 0 if and only if X is contractible.

2.  $cat(S^n) = 1$ .

3. More generally, cat(X) = 1 if and only if X is a nontrivial co-H space.

4.  $cat(T^n) = n$  (this follows from the proposition and property (3) above).

We single out an example of interest in dynamical systems which, although quite simple, does not seem to be well known among dynamicists. (The analogue for Kähler manifolds *is* well known among topologists.) 5. If  $M^{2n}$  is a simply connected compact symplectic manifold, then  $\operatorname{cat}(M) = n = \frac{1}{2} \operatorname{dim}(M)$ . (First, observe that the volume form is not exact since it represents a nontrivial fundamental class of M. Because  $\omega^n/n! = \operatorname{vol}(\operatorname{see}[1], p. 165)$ , the nondegenerate closed 2-form  $\omega$  cannot be exact either. Hence,  $\omega^n$  represents a nontrivial cup-product of length n in **R**-cohomology. By property (4) above,  $\operatorname{cat}(M) \leq (\dim M)/2 = n$ . Hence,

$$n \leq \operatorname{cup}(M) \leq \operatorname{cat}(M) \leq \frac{1}{2} \dim M = n$$

and the result follows.)

## §2. RATIONAL HOMOTOPY AND CATEGORY

The basic reference for this section is [3]. To each space X, Sullivan functorially associated a commutative differential graded algebra (A(X), d) of rational polynomial forms possessing the salient property that integration defines a natural algebra isomorphism between  $H^*(A(X), d)$  and  $H^*(X; \mathbf{Q})$ . Furthermore, the cdga A(X) was shown to contain all the rational homotopy information about X; information which may be gleaned from an associated cdga *minimal model* of A(X).

A cdga  $(\Lambda, d)$  is minimal if (1)  $\Lambda = \Lambda X$ , where  $X = \bigoplus_{i>0} X^i$  is a graded Q-vector space and  $\Lambda X$  denotes that  $\Lambda$  is freely generated by X; that is,  $\Lambda X =$  Symmetric algebra  $(X^{\text{even}}) \otimes$  Exterior algebra  $(X^{\text{odd}})$ . (2) There is a basis for  $X, \{x_{\alpha}\}_{\alpha \in I}$ , so that if I is well ordered by <, then  $dx_{\beta} \in \Lambda_{\alpha<\beta}^+(x_{\alpha})$  $\cdot \Lambda_{\alpha<\beta}^+(x_{\alpha})$ . That is,  $\Lambda$  is constructed by stages and the differentials of  $\beta^{\text{th}}$ stage generators are decomposable in the generators of previous stages.

A minimal model for a space M is a minimal cdga  $\Lambda(M)$  and a cdga map  $\Lambda(M) \rightarrow A(M)$  inducing an isomorphism in cohomology. The fundamental theorem of rational homotopy theory is then (see [4] for example).

THEOREM. Each space M has a minimal model  $\Lambda(M)$  and, furthermore, for nilpotent spaces the stage by stage construction precisely mirrors the rational Postnikov tower with the differential corresponding to the k-invariant.

Recall that a space M is *nilpotent* if its fundamental group  $\pi_1(M)$  is a nilpotent group and the natural action of  $\pi_1(M)$  on  $\pi_n(M)$  (see [10]) is a nilpotent action (see [12]). In particular, any simply connected space or any  $K(\pi, 1)$  with  $\pi$  nilpotent is a nilpotent space. The theorem then says that, for