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## TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

by Jean-Paul ALLOUCHE and Roland BACHER

**ABSTRACT.** What is the relationship between folding a piece of paper, moving disks in the classical tower of Hanoi algorithm and searching for minimal sequences of integers having no  $p$  terms in arithmetic progression? Our aim is to show how the Toeplitz sequences introduced by Jacobs and Keane in [15] allow us to give (*inter alia*) a unified description of the preceding problems. We give moreover some connections between Toeplitz sequences and  $q$ -automatic sequences.

### 1. TOEPLITZ SEQUENCES

In [15], (see also [21]), Jacobs and Keane defined the notion of Toeplitz sequence: they wanted to construct “explicit” sequences giving rise to strictly ergodic systems. They proved moreover that the unique invariant measure attached to such a sequence has a discrete rational spectrum. Roughly speaking a Toeplitz sequence is obtained by successive insertions of periodic sequences into the “holes” of a given periodic sequence, (a precise definition is given below). This construction was inspired by a device used by Toeplitz [28] for building explicitly almost periodic real functions. The method of Jacobs and Keane has since been used by many people working in ergodic theory (see for instance [29], [16] and [25], see also [14] and its impressive bibliography). We now give the definition of a Toeplitz sequence (compare with [15], [16], [14] and [29]):

Let  $\Gamma = \{a_1, \dots, a_r, \omega\}$  be an alphabet (finite set) with a “marked” letter (“hole”)  $\omega$ . If  $B = (B(k))_{k \geq 0}$  is a sequence with values in  $\Gamma$ , we define a transformation  $T_B: \Gamma^{\mathbb{N}} \rightarrow \Gamma^{\mathbb{N}}$  as follows: for any sequence  $C = (C(k))_{k \geq 0}$  with values in  $\Gamma$ , let  $h_0 < h_1 < \dots$  be the increasing sequence (which might

be finite or even empty) of those integers  $h$  for which  $C(h) = \omega$ . Then one defines

$$\begin{aligned} T_B C(j) &= C(j) \quad \text{if} \quad C(j) \neq \omega, \\ T_B C(h_k) &= B(k) \quad \text{for every } k. \end{aligned}$$

Suppose we are now given a sequence of periodic sequences  $B_0, B_1, \dots, B_k, \dots$  with values in  $\Gamma$ , and such that the zeroth value of each  $B_j$  is not equal to  $\omega$ . Writing  $T_j$  instead of  $T_{B_j}$ , we then define a sequence of periodic sequences as follows:

$$\begin{aligned} A_0 &= B_0 \\ A_1 &= T_1(A_0) \\ A_2 &= T_2(A_1) = T_2(T_1(A_0)) \\ &\dots \\ A_{k+1} &= T_{k+1}(A_k) = T_{k+1}(T_k(\dots(T_1(A_0))\dots)). \end{aligned}$$

As  $k$  goes to infinity the sequence  $A_k$  tends to a limit  $A$  with values in  $\Gamma - \{\omega\}$  (the existence of this limit, for the topology of simple convergence, is left to the reader): such a sequence is called a *Toeplitz sequence*.

An alternative (equivalent) definition of a Toeplitz sequence is given in [29]:

$A$  is a Toeplitz sequence if and only if one has

$$\forall n \in \mathbb{N} \quad \exists p \in \mathbb{N}^* \quad \forall n' \equiv n \pmod{p} \quad A(n') = A(n).$$

In what follows we first suppose that the set  $\Gamma$  is not necessarily a finite set; second, we restrict ourselves to the case where the sequence  $B_0, B_1, \dots$  has the following form: there exist a periodic sequence  $B$  with values in  $\Gamma$  such that  $B(0) \neq \omega$  and a function  $f$  from  $\Gamma$  to  $\Gamma$  with  $f^{-1}(\omega) = \{\omega\}$ , such that

$$\forall k \geq 0 \quad B_k = f^{(k)}(B),$$

where  $f^{(k)}$  is the  $k^{\text{th}}$  iterate of the function  $f$  and  $f^{(k)}(B)$  is the termwise image of the sequence  $B$  under  $f^{(k)}$ ; the resulting Toeplitz sequence

$$A = \lim_{k \rightarrow \infty} T_k(\dots T_2(T_1(B))\dots)$$

$$(\text{where } T_k = T_{B_k} = T_{f^{(k)}(B)})$$

will be called the *Toeplitz transform* of  $(B, f)$  and denoted by  $Tt(B, f)$ .

*Example.* Let  $B$  be the sequence  $(0\omega 1\omega)^\infty$ , and let  $f$  be defined on  $\{0, 1, \omega\}$  by  $f(0) = 1$ ,  $f(1) = 0$ ,  $f(\omega) = \omega$ , then one has:

$$B_0 = (0\omega 1\omega 0\omega 1\omega \cdots) (= B),$$

$$B_1 = (1\omega 0\omega 1\omega 0\omega \cdots),$$

$$B_2 = B_0,$$

...

$$A_0 = (0\omega 1\omega 0\omega 1\omega \cdots),$$

$$A_1 = (011\omega 001\omega \cdots),$$

$$A_2 = (0110001\omega \cdots),$$

...

Note that if  $\Gamma$  is finite, and  $f$  one-to-one, such a sequence  $Tt(B, f)$  can also be obtained by replacing  $B$  by a sequence of greater period and  $f$  by  $id$ .

We now give four examples of Toeplitz transforms in (apparently) unrelated domains.

## 2. PAPERFOLDING SEQUENCES AND TOEPLITZ TRANSFORMS

In [23] and [22] Prodinger and Urbanek study the Toeplitz transform of  $((0\omega 1\omega)^\infty, id)$  and of  $((0\omega 1\omega 1\omega 0\omega)^\infty, id)$ . They prove that these sequences do not have arbitrarily long squares (a sequence  $A$  contains a square of length  $2k$  if there exists an index  $j$  such that  $A(j+n) = A(j+n+k)$  for every  $n$  between 0 and  $k-1$ ). Dekking already noticed in [10] that the first sequence is nothing but the regular paperfolding sequence (see [9], [18], [20], [17]), which is obtained by repeatedly folding a piece of paper, and we obtained in [1] the same result as Prodinger and Urbanek for the general paperfolding sequences. Let us give here two simple examples:

**PROPOSITION.** *Let  $B$  be the sequence  $B = (0\omega 1\omega)^\infty$  and let  $f$  be defined by  $f(0) = 1$ ,  $f(1) = 0$  and  $f(\omega) = \omega$ . Then*

*the sequence  $Tt(B, id)$  is the regular paperfolding sequence,  
the sequence  $Tt(B, f)$  is the alternate paperfolding sequence.*