1. Toeplitz sequences

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TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

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ABSTRACT. What is the relationship between folding a piece of paper, moving disks in the classical tower of Hanoi algorithm and searching for minimal sequences of integers having no p terms in arithmetic progression? Our aim is to show how the Toeplitz sequences introduced by Jacobs and Keane in [15] allow us to give *(inter alia)* a unified description of the preceding problems. We give moreover some connections between Toeplitz sequences and q-automatic sequences.

1. TOEPLITZ SEQUENCES

In [15], (see also [21]), Jacobs and Keane defined the notion of Toeplitz sequence: they wanted to construct "explicit" sequences giving rise to strictly ergodic systems. They proved moreover that the unique invariant measure attached to such a sequence has a discrete rational spectrum. Roughly speaking a Toeplitz sequence is obtained by successive insertions of periodic sequences into the "holes" of a given periodic sequence, (a precise definition is given below). This construction was inspired by a device used by Toeplitz [28] for building explicitly almost periodic real functions. The method of Jacobs and Keane has since been used by many people working in ergodic theory (see for instance [29], [16] and [25], see also [14] and its impressive bibliography). We now give the definition of a Toeplitz sequence (compare with [15], [16], [14] and [29]):

Let $\Gamma = \{a_1, \cdots, a_r, \omega\}$ be an alphabet (finite set) with a "marked" letter ("hole") ω. If $B = (B(k))_{k \geqslant 0}$ is a sequence with values in Γ, we define a transformation $T_B: \Gamma^{\mathbf{N}} \to \Gamma^{\mathbf{N}}$ as follows: for any sequence $C = (C(k))_{k \geqslant 0}$ with values in Γ, let $h_0 < h_1 < \cdots$ be the increasing sequence (which might be finite or even empty) of those integers h for which $C(h) = \omega$. Then one defines

$$T_B C(j) = C(j) \quad \text{if} \quad C(j) \neq \omega\ ,$$
$$T_B C(h_k) = B(k) \quad \text{for every } k\ .$$

Suppose we are now given a sequence of periodic sequences $B_0, B_1, \cdots, B_k, \cdots$ with values in Γ, and such that the zeroth value of each B_j is not equal to ω. Writing T_j instead of T_{B_j}, we then define a sequence of periodic sequences as follows:

$$\begin{aligned}
A_0 &= B_0 \\
A_1 &= T_1(A_0) \\
A_2 &= T_2(A_1) \quad = T_2(T_1(A_0)) \\
&\cdots \\
A_{k+1} &= T_{k+1}(A_k) = T_{k+1}(T_k(\cdots(T_1(A_0))\cdots))\ .
\end{aligned}$$

As k goes to infinity the sequence A_k tends to a limit A with values in $\Gamma - \{\omega\}$ (the existence of this limit, for the topology of simple convergence, is left to the reader): such a sequence is called a *Toeplitz sequence.*
An alternative (equivalent) definition of a Toeplitz sequence is given in [29]:

A is a Toeplitz sequence if and only if one has

$$\forall n \in \mathbf{N} \quad \exists p \in \mathbf{N}^* \quad \forall n' \equiv n \,(\mathrm{mod}\, p) \quad A(n') = A(n)\ .$$

In what follows we first suppose that the set Γ is not necessarily a finite set; second, we restrict ourselves to the case where the sequence $B_0, B_1, \cdots$ has the following form: there exist a periodic sequence B with values in Γ such that $B(0) \neq \omega$ and a function f from Γ to Γ with $f^{-1}(\omega) = \{\omega\}$, such that

$$\forall k \geqslant 0 \quad B_k = f^{(k)}(B)\ ,$$

where $f^{(k)}$ is the k^{th} iterate of the function f and $f^{(k)}(B)$ is the termwise image of the sequence B under $f^{(k)}$; the resulting Toeplitz sequence

$$A = \lim_{k \to \infty} T_k(\cdots T_2(T_1(B))\cdots)$$

$$(\text{where } T_k = T_{B_k} = T_{f^{(k)}(B)})$$

will be called the *Toeplitz transform* of (B, f) and denoted by $Tt(B, f)$.

Example. Let B be the sequence $(0\omega 1\omega)^\infty$, and let f be defined on $\{0, 1, \omega\}$ by $f(0) = 1$, $f(1) = 0$, $f(\omega) = \omega$, then one has:

$$
\begin{aligned}
B_0 &= (0\omega 1\omega 0\omega 1\omega \cdots)\ (= B)\ , \\
B_1 &= (1\omega 0\omega 1\omega 0\omega \cdots)\ , \\
B_2 &= B_0\ , \\
&\ldots \\
A_0 &= (0\omega 1\omega 0\omega 1\omega \cdots)\ , \\
A_1 &= (011\omega 001\omega \cdots)\ , \\
A_2 &= (0110001\omega \cdots)\ , \\
&\ldots .
\end{aligned}
$$

Note that if Γ is finite, and f one-to-one, such a sequence $Tt(B,f)$ can also be obtained by replacing B by a sequence of greater period and f by *id*.

We now give four examples of Toeplitz transforms in (apparently) unrelated domains.

2. Paperfolding sequences and Toeplitz transforms

In [23] and [22] Prodinger and Urbanek study the Toeplitz transform of $((0\omega 1\omega)^\infty, id)$ and of $((0\omega 1\omega 1\omega 0\omega)^\infty, id)$. They prove that these sequences do not have arbitrarily long squares (a sequence A contains a square of length $2k$ if there exists an index j such that $A(j+n) = A(j+n+k)$ for every n between 0 and $k-1$). Dekking already noticed in [10] that the first sequence is nothing but the regular paperfolding sequence (see [9], [18], [20], [17]), which is obtained by repeatedly folding a piece of paper, and we obtained in [1] the same result as Prodinger and Urbanek for the general paperfolding sequences. Let us give here two simple examples:

PROPOSITION. *Let B be the sequence $B = (0\omega 1\omega)^\infty$ and let f be defined by $f(0) = 1$, $f(1) = 0$ and $f(\omega) = \omega$. Then*

the sequence $Tt(B, id)$ is the regular paperfolding sequence,

the sequence $Tt(B, f)$ is the alternate paperfolding sequence.